MODAL LOGIC NOTES

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These are some notes on propositional modal logic that are partially based on a seminar given at the University of Notre Dame by Tim Bays in Spring. In turn this seminar was partially based on the textbook *First Steps in Modal Logic* by Sally Popkorn, a pseudonym of the late mathematician Harrold Simmons. I plan on expanding on them soon. Nothing new here. Mistakes are my own.

1. Review of Propositional Logic

1.1. **Propositional Languages.**

Definition 1.1. *Let* Φ *be a countable set of* propositional variables. *The set of* propositional formulas, $F(\Phi)$, *is the smallest set such that:*

(i) $\Phi \subseteq F(\Phi)$;
(ii) $\bot, \top \in F(\Phi)$;
(iii) $\neg\varphi \in F(\Phi)$ *if* $\varphi \in F(\Phi)$;
(iv) $(\varphi \wedge \psi) \in F(\Phi)$ *if* $\varphi \in F(\Phi)$ *and* $\psi \in F(\Phi)$.

We will use the following abbreviations:

(1) $(\varphi \vee \psi) \coloneqq \neg(\neg\varphi \wedge \neg\psi)$
(2) $(\varphi \to \psi) \coloneqq \neg\varphi \vee \psi$
(3) $(\varphi \leftrightarrow \psi) \coloneqq (\varphi \to \psi) \wedge (\psi \to \varphi)$

The restriction to a countable set of propositional variables ensures that $F(\Phi)$ is itself a countable set. The restriction is perhaps a bit arbitrary but it simplifies slightly some of the proofs given below.

From here on out we'll fix a set of formulas $F(\Phi)$. By 'formula' we simply mean and element of $F(\Phi)$ and by a propositional variable we mean an element of Φ.

Definition 1.2. *The set of* atomic formulas *is* $\Phi \cup \{\bot, \top\}$.

1.2. **Valuations.**

Definition 1.3. *A* valuation *is a map* $v : \Phi \to 2$ *(where* $2 = \{0, 1\}$*).*

Any valuation $v : \Phi \to 2$ *uniquely extends to a map*

$$v^* : F(\Phi) \to 2$$

with the property that

- $v^*(p) = v(p)$ *for* $p \in \Phi$*;*
- $v^*(\bot) = 0$ *and* $v^*(\top) = 1$*;*
- $v^*(\varphi \wedge \psi) = \min(v^*(\varphi), v^*(\psi))$

Definition 1.4. *We say that a valuation* v satisfies *a formula* φ*, written* $v \vDash \varphi$*, if* $v^*(\varphi) = 1$*. We say that* v *satisfies a set* Γ *of formulas if* $v \vDash \varphi$ *for each* $\varphi \in \Gamma$*.*

Proposition 1.5. *Let* v *be a valuation. Then*

(i) $v \vDash p$ *if and only if* $v(p) = 1$*, for* $p \in \Phi$*;*
(ii) $v \vDash \top$ *and* $v \nvDash \bot$*;*
(iii) $v \vDash \neg\varphi$ *if and only if* $v \nvDash \varphi$*;*
(iv) $v \vDash (\varphi \wedge \psi)$ *if and only if* $v \vDash \varphi$ *and* $v \vDash \psi$*;*
(v) $v \vDash (\varphi \vee \psi)$ *if and only if* $v \vDash \varphi$ *or* $v \vDash \psi$*;*
(vi) $v \vDash (\varphi \to \psi)$ *if and only if* $v \nvDash \varphi$ *or* $v \vDash \psi$*;*

Definition 1.6. *Let* Γ *be a set of formulas and* φ *and formula. We say that* Γ entails φ*, or that* φ *is a* consequence of Γ, $\Gamma \vDash \varphi$*, if for every valuation* v*,* $v \vDash \Gamma$ *only if* $v \vDash \varphi$*.*

Definition 1.7. *A formula* φ *is called a* tautology *if* $\emptyset \vDash \varphi$ *(i.e.,* $v^*(\varphi) = 1$ *for every valuation* v*).*

1.3. **Compactness.**

Definition 1.8. *A set of formulas* Γ *is* satisfiable *if* $v \vDash \Gamma$ *for some valuation* v*.*

Definition 1.9. *A set of formulas* Γ *is* finitely satisfiable *if every finite subset of* Γ *is satisfiable.*

Remark 1.10. *Note that if* Γ *is satisfiable, then it is also finitely satisfiable.*

Definition 1.11. *A set of formulas* Γ *is* maximally finitely satisfiable *if*

(1) Γ *is finitely satisfiable, and*

(2) for every φ*, either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*.*

Lemma 1.12. *Let* Γ *be finitely satisfiable and let* φ *be a formula. Then either*

(1) $\Gamma \cup \{\varphi\}$ *is finitely satisfiable, or*

(2) $\Gamma \cup \{\neg\varphi\}$ *is finitely satisfiable.*

Proof. Suppose not. Then since $\Gamma \cup \{\varphi\}$ is not finitely satisfiable there is a finite subset X of Γ such that $X \cup \{\varphi\}$ is not satisfiable. Similarly, since $\Gamma \cup \{\neg\varphi\}$ is not finitely satisfiable, there is a finite subset of Y of Γ such that $Y \cup \{\neg\varphi\}$ is not satisfiable. Now note that since X and Y are both finite subsets of Γ, their union $X \cup Y$ is also a finite subset of Γ. And so since Γ is finitely satisfiable, there is some valuation v such that $v \vDash X \cup Y$. Now since $v \vDash \neg\varphi$ if and only if $v \nvDash \varphi$, we have that *either* $v \vDash \varphi$ or $v \vDash \neg\varphi$. But if $v \vDash \varphi$, then $v \vDash X \cup \{\varphi\}$, contradicting the fact that $X \cup \{\varphi\}$ is not satisfiable. And if $v \vDash \neg\varphi$, then $v \vDash Y \cup \{\neg\varphi\}$, contradicting the fact that $Y \cup \{\neg\varphi\}$ is not finitely satisfiable. Either way we get a contradiction and so the lemma is proved. ☐

Lemma 1.13. *Let* Γ *be finitely satisfiable. Then there exists* Γ' *such that*

(1) $\Gamma \subseteq \Gamma'$*, and*

(2) Γ' *is maximally finitely satisfiable.*

Proof. Let $\varphi_1, \varphi_2, \ldots$ be a list of all of the formulas of our language (note that there is such a list because we assumed that Φ was countable!). We define a sequence of sets of formulas $\Gamma_0, \Gamma_1, \ldots$ by

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_{n+1}\} \text{ if finitely satisfiable} \\ \Gamma_n \cup \{\neg\varphi_{n+1}\}\text{otherwise} \end{cases}$$

Let $\Gamma' = \bigcup_{n<\omega} \Gamma_n$. Clearly $\Gamma \subseteq \Gamma'$. We claim that Γ' is also maximally finitely satisfiable. To see this let X be a finite subset of Γ'. Then by construction there is some n such that X is a finite subset of Γ_n. Now by the previous lemma and the construction of Γ_n, we know that Γ_n is finitely satisfiable. Hence X is satisfiable. Thus Γ' is finitely satisfiable. Finally, to show that it is maximal note that for any formula φ, there is some n such that $\varphi = \varphi_n$. Then $\varphi_n \in \Gamma_n$ or $\neg\varphi_n \in \Gamma_n$. Hence Γ' is maximally finitely satisfiable. □

Lemma 1.14. *Let Γ be maximally finitely satisfiable. Then Γ is satisfiable.*

Proof. Let v be the valuation such that $v(p) = 1$ if and only if $p \in \Gamma$. We show by induction that $v \vDash \varphi$ if and only if $\varphi \in \Gamma$.

If p is a propositional variable we have

$$\begin{aligned} v \vDash p &\iff v(p) = 1 \\ &\iff p \in \Gamma \end{aligned}$$

Since Γ is maximally finitely satisfiable, $\bot \notin \Gamma$ and $\top \in \Gamma$.

Now suppose that φ has the desired property. Then

$$\begin{aligned} v \vDash \neg\varphi &\iff v \nvDash \varphi \\ &\iff \varphi \notin \Gamma \\ &\iff \neg\varphi \in \Gamma \end{aligned}$$

The first equivalence follows by definition, the second by induction, and the third by the fact that Γ is maximally finitely satisfiable. Now suppose that φ and ψ have the desired property. Then

$$\begin{aligned} v \vDash (\varphi \wedge \psi) &\iff v \vDash \varphi \text{ and } v \vDash \psi \\ &\iff \varphi \in \Gamma \text{ and } \psi \in \Gamma \\ &\iff (\varphi \wedge \psi) \in \Gamma \end{aligned}$$

The first equivalence follows by definition, the second by the induction hypothesis, and the last by the fact that Γ is maximally finitely satisfiable. □

Theorem 1.15 (Compactness)**.** *If Γ is finitely satisfiable then Γ is satisfiable.*

Proof. Let Γ be finitely satisfiable. Then by a lemma there exists Γ' such that $\Gamma \subseteq \Gamma'$ and Γ' is maximally finitely satisfiable. By our last lemma Γ' is satisfiable. Hence Γ is satisfiable (by the same valuation). □

2. Modal Logic

2.1. **The modal language.**

Definition 2.1. *Let Φ be a countable collection of propositional variables and I an* index set*. The set $M(\Phi, I)$ of* modal formulas *indexed by I is the smallest set such that*

- $\Phi \subseteq F(\Phi, I)$
- $\bot, \top \in F(\Phi, I)$
- $\neg\varphi \in F(\Phi, I)$ *if* $\varphi \in F(\Phi, I)$
- $(\varphi \wedge \psi) \in F(\Phi, I)$ *if* $\varphi, \psi \in F(\Phi, I)$
- $\Box_i\varphi \in F(\Phi, I)$ *if* $\varphi \in F(\Phi, I)$ *and* $i \in I$.

More often than not we will be focused on modal languages with a single modal operator. When we do so we will simply write $\Box$ without any subscript. Throughout we will take $\Diamond_i\varphi$ as an abbreviation for $\neg\Box_i\neg\varphi$.

Note that since the modal formulas are defined by induction, we can still give proofs by induction. But we have to add one further clause: namely that if φ has a certain property X, so does $\Box_i\varphi$, for each $i \in I$.

2.2. **Frames, valuations, and satisfaction.** In what follows we will suppose that some set of modal formulas is given with modal operators indexed by the set I.

Definition 2.2. *Let I be an index set. An* I-frame *is a pair (W, R) such that W is a non-empty set and $R : I \to \mathcal{P}(W \times W)$ is a function from I to binary relations $R(i)$ on W.*

We will write $R(i)$ as R_i for readability. When I is a singleton we simply write R for R_i where i is the unique element of I.

Definition 2.3. *An* I- model *is a triple (W, R, V) such that (W, R) is an I-frame and and $V : \Phi \to \mathcal{P}(W)$ is a map from propositional variables to subsets of W. We call the map V a* valuation*. Given a model $\mathcal{M} = (W, R, V)$, we say that $\mathcal{M}$ is* based on *on (W, R), or that (W, R) is the frame* underlying $\mathcal{M}$

In what follows we will often speak simply of "frames" and "models" leaving reference to the index set implicit when no confusion will arise.

Definition 2.4. *Let w be a world in the model $\mathcal{M} = (W, RV)$. Then we inductively define the notion of a formula φ being* satisfied *(or* true*) in $\mathcal{M}$ at world w as follows:*

(1) $\mathcal{M}, w \Vdash p$ if and only if $w \in V(p)$, where $p \in \Phi$
(2) $\mathcal{M}, w \Vdash \top$
(3) $\mathcal{M}, w \Vdash \neg\varphi$ if and only if $\mathcal{M}, w \nVdash \varphi$
(4) $\mathcal{M}, w \Vdash (\varphi \wedge \psi)$ if and only if $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$,
(5) $\mathcal{M}, v \Vdash \Box_i\varphi$ if and only if for all $v \in W$ with R_iwv we have $\mathcal{M}, v \Vdash \varphi$.

We say that a set of formulas Γ is true at w in model $\mathcal{M}$ if $\mathcal{M}, w \Vdash \varphi$ for each $\varphi \in \Gamma$.

It follows by omission that $\mathcal{M}, w \nVdash \bot$. When no ambiguity arises we will simply write $w \Vdash \varphi$ for $\mathcal{M}, w \Vdash \varphi$.

Remark 2.5. *I've written R_iwv for $(w,v) \in R_i$. There are a couple other notational variants I'm going to use for this. First, note that binary relations on a set W are in one to one correspondence with function $W \to \mathcal{P}(W)$. Thus where R is a binary relation on W, I will sometimes write $R(w)$ for the set of all v such that Rwv. Hence $v \in R(w)$ if and only if Rwv. Finally, to emphasize the perspective that the worlds of a frame are states and the accessibility relations representing possible transitions between states, I will sometimes write $w \xrightarrow{R_i} v$ for R_iwv.*

2.3. **Structural Characterizations.** We can describe some properties of worlds in a model in terms of which modal formulas are true is that model. Suppose our language contains a single modal operator $\Box$ and consider a model $\mathcal{M} = (W, R, V)$. We say that w *sees* v if Rwv. We say that w is *blind* if it doesn't see any world. Then the following are not hard to verify

Example 2.6. *(1) $w \Vdash \Box\bot$ if and only if w is blind.*

(2) $w \Vdash \Diamond\top$ if and only if w is not blind.

(3) $w \Vdash \Box\Box\bot$ if and only if every world that w sees is blind.

2.4. **More Satisfaction Relations.**

Definition 2.7. *We say that φ is* true *in a model $\mathcal{M}$, written $\mathcal{M} \Vdash \varphi$, if φ is true at every world in $\mathcal{M}$ (i.e., $\mathcal{M}, w \Vdash \varphi$ for all $w \in W$).*

A set Γ *of formulas is true in $\mathcal{M}$ if $\mathcal{M}, w \Vdash \Gamma$ for all $\varphi \in \Gamma$.*

Definition 2.8. *A formula φ is* valid on a frame *$\mathcal{F}$, written $\mathcal{F} \Vdash \varphi$, if it is true in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$ (i.e., $(\mathcal{F}, V) \Vdash \varphi$ for ever valuation V).*

A formula φ is valid on a class of frames *K, written $\mathsf{K} \Vdash \varphi$ if φ is valid on every frame in K.*

Remark 2.9. *There is one further notion we could define that will play less of a role in what follows. Consider a frame $\mathcal{F} = (W, R)$. We could then define the notion of a formula*

φ being valid at a point $w \in W$*,* $\mathcal{F}, w \Vdash \varphi$*, by stating that* φ *is true at* w *in any model based on* $\mathcal{F}$ *(i.e.,* $\mathcal{M}, w \Vdash \varphi$ *for any model* $\mathcal{M} = (W, R_i, V)$*. I'm unsure what significance such a notion has. Perhaps it would be more useful if our interest were in* pointed frames*: frames with a distinguished point.*

Lemma 2.10. *Let* $\mathcal{F}$ *be a frame. Then* $\mathcal{F} \Vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Proof. Let $\mathcal{M} = (\mathcal{F}, V)$ be a model and $w \in W$. Suppose that $w \Vdash \Box(\varphi \to \psi)$ and $w \Vdash \Box\varphi$. Let $v \in W$ be such that Rwv. Then since $w \Vdash \Box(\varphi \to \psi)$, $v \Vdash \varphi \to \psi$. And since $w \Vdash \Box\varphi$, $v \Vdash \varphi$. Hence $v \Vdash \psi$. So since v was chosen arbitrarily, $w \Vdash \Box\psi$. Therefore $w \Vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. □

Lemma 2.11. *Let* $\mathcal{M}$ *be a model. Then* $\mathcal{M} \Vdash \varphi$ *only if* $\mathcal{M} \Vdash \Box\varphi$*.*

Proof. Suppose that $\mathcal{M} \Vdash \varphi$. Then for each $w \in W$, $\mathcal{M}, w \Vdash \varphi$. Let $w \in W$ and wRv be arbitrary. Then since $\mathcal{M} \Vdash \varphi$, $\mathcal{M}, v \Vdash \varphi$. Since v was arbitrary, $w \Vdash \Box\varphi$. Since w was arbitrary $\mathcal{M} \Vdash \Box\varphi$. □

Corollary 2.12. *Let* $\mathcal{F}$ *be a frame. Then* $\mathcal{F} \Vdash \varphi$ *only if* $\mathcal{F} \Vdash \Box\varphi$*.*

Proof. Suppose that $\mathcal{F} \Vdash \varphi$. Let V be any valuation. Then $(\mathcal{F}, V) \Vdash \varphi$. By our last lemma $(\mathcal{F}, V) \Vdash \Box\varphi$. Since V was arbitrary, $\mathcal{F} \Vdash \Box\varphi$. □

The following lemma will be proved later. And while we will rely on this lemma going forward, no circularity is induced because the machinery we will use to later prove the lemma does not rely on any theorems developed in terms of the lemma. The reason to put it off is that in order to prove we need to diverge from our current trajectory a bit in a way that would be perhaps a bit distracting.

Lemma 2.13. *Suppose that* ψ *is a substitution instance of* φ*. Then* $\mathcal{F} \Vdash \varphi$ *only if* $\mathcal{F} \Vdash \psi$*.*

Lemma 2.14. *Let* φ *be a propositional tautology. Then for any model* $\mathcal{M}$*,* $\mathcal{M} \Vdash \varphi$*.*

Proof. Let $\mathcal{M} = (W, R_i, V)$ be a model. Then for each $w \in W$ we define a map $v : \Phi \to 2$ by $v_w(p) = 1$ if and only if $w \in V(p)$. By a trivial induction, we have $v_w \vDash \varphi$ if and only if $w \Vdash \varphi$, for any non-modal formula φ. So since $v_w \Vdash \varphi$ for any propositional tautology, $w \Vdash \varphi$. Since w was chosen arbitrarily we then have that $\mathcal{M} \Vdash \varphi$ for any propositional tautology φ. □

Corollary 2.15. *Let ψ be a substitution instance of a propositional tautology. Then for any frame $\mathcal{F}$, we have $\mathcal{F} \Vdash \varphi$.*

Proof. Suppose that ψ is a substitution instance of a propositional tautology φ. By our last lemma, $\mathcal{F} \Vdash \varphi$. By an older lemma, it then follows that $\mathcal{F} \Vdash \psi$. □

3. Correspondence Theory

Correspondence theory studies the relationship between structural characterizations of the accessibility relation and the modal formulas valid on frames. In a sense we've already seen once such result: the modal formula $\Box(\varphi \to \psi) \to (\Box\varphi \to \varphi\psi)$ is valid on every frame and so corresponds to the requirement on a accessibility relation R that, for instance, $R(x, y) \to R(x, y)$. This section looks at more interesting results.

3.1. **Some elementary correspondence results.**

Proposition 3.1. *Let $\mathcal{F} = (W, R)$ be a frame. Then the following are equivalent.*

(1) $\mathcal{F} \Vdash \Box p \to \Box\Box p$

(2) $\mathcal{F} \Vdash \Box\varphi \to \Box\Box\varphi$

(3) R is transitive

Proof. Trivially (2) implies (1). (1) implies (2) since $\Box\varphi \to \Box\Box\varphi$ is a substitution instance of $\Box p \to \Box\Box p$ and, by a previous lemma, every substitution instance of a formula valid on a frame is valid on that frame.

To show that (1) implies (3) suppose that $\mathcal{F} \Vdash \Box p \to \Box\Box p$. Let $x, y, z \in W$ be such that Rxy and Ryz. Let V be a valuation that maps p to the set of worlds that x sees (i.e.,

$V(p) = \{w \mid Rxw\}$. Then for all w such that Rxw, $w \in V(p)$. Hence $x \Vdash \Box p$. Since $\mathcal{F} \Vdash \Box p \to \Box\Box p$ it follows that $x \Vdash \Box p \to \Box\Box p$. Thus $x \Vdash \Box\Box p$. Since Rxy it follows that $y \Vdash \Box p$. And since Ryz it follows that $z \Vdash P$. So $z \in V(P)$. By the definition of V it then follows that Rxz. Hence R is transitive.

To show that (3) implies (1) suppose now that R is transitive. Let $\mathcal{M}$ and $x \in W$ be an arbitrary model and point in that model such that $x \Vdash \Box P$. Suppose for reductio that $x \nVdash \Box\Box p$. Thus there is some y with Rxy such that $y \nVdash \Box p$. Thus there is some z with Ryz such that $z \nVdash p$. Since R is transtive, Rxz. This contradicts the fact that $x \Vdash \Box p$. So $x \Vdash \Box\Box p$. So $x \Vdash \Box p \to \Box\Box p$.

□

Proposition 3.2. *The following are equivalent:*

(1) $\mathcal{F} \Vdash \Box p \to p$.

(2) $\mathcal{F} \Vdash \Box\varphi \to \varphi$.

(3) R *is reflexive.*

Proof. (1) and (2) are equivalent by the same reasoning as above.

To show that (1) implies (3), suppose that $\mathcal{F} \Vdash \Box p \to p$. Let $w \in W$ be arbitrary and define a valuation V so that $V(p) = \{x \mid Rwx\}$. Then $w \Vdash \Box p$. And so since $\mathcal{F} \Vdash \Box p \to p$ we have $w \Vdash p$. By the definition of our valuation Rww.

To show that (3) implies (1), suppose that R is reflexive. Let $\mathcal{M}$ be an arbitrary model based on $\mathcal{F}$ and let $w \in W$ be such that $w \Vdash \Box p$. Since R is reflexive, Rww. So $w \Vdash p$. Hence, $w \Vdash \Box p \to p$ as desired. □

Proposition 3.3. *The following are equivalent:*

(1) $\mathcal{F} \Vdash p \to \Box\Diamond p$.

(2) $\mathcal{F} \Vdash \varphi \to \Box\Diamond\varphi$.

(3) R *is symmetric.*

Proof. (1) and (2) are equivalent by the same reasoning as above.

To show that (1) implies (3), suppose that $\mathcal{F} \Vdash p \to \Box\Diamond p$. Let w and v be arbitrary worlds such that Rwv. Define a valuation V such that $V(p) = \{w\}$. Then $w \Vdash p$. Since $\mathcal{F} \Vdash p \to \Box\Diamond p$, it then follows that $w \Vdash \Box\Diamond p$. Since Rwv, $v \Vdash \Diamond p$. So there exists some x such that Rvx and $x \Vdash P$. By the definition of our valuation, $x = w$. Thus Rvw and so R is symmetric.

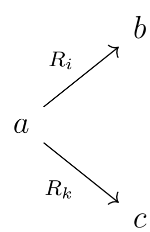
To show that (3) implies (1) suppose that R is symmetric. Let $\mathcal{M}$ be an arbitary model based on $\mathcal{F}$ and $w \in W$ such that $w \Vdash p$. Let v be an arbitrary world with Rwv. Since R is symmetric, it follows that Rvw. So since $w \Vdash p$, it follows that $v \Vdash \Diamond p$. Since v was chosen arbitrarily, it follows that $w \Vdash \Box\Diamond p$. Hence $w \Vdash p \to \Box\Diamond p$. So $\mathcal{F} \Vdash p \to \Box\Diamond p$. □

Theorem 3.4 (Confluence)**.** *The following are equivalent:*

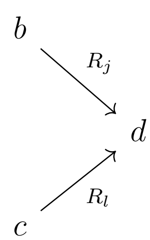
(1) $\mathcal{F} \Vdash \Diamond_i\Box_j p \to \Box_k\Diamond_l p$

(2) $\mathcal{F} \Vdash \Diamond_i\Box_j \varphi \to \Box_k\Diamond_l \varphi$

(3) $\mathcal{F}$ *has the confluence property. That is, for any wedge*



there exists a $d \in W$ *such that*



Proof. (1) and (2) are equivalent by the same reasoning as above.

To show that (1) implies (3) suppose that $\mathcal{F} \Vdash \Diamond_i\Box_j p \to \Box_k\Diamond_l p$. Let a, b and c be arbitrary worlds such that $R_i ab$ and $R_k ac$. Define a valuation V by $v(p) = \{x \mid R_j bx\}$. Then $b \Vdash \Box_j p$ and so $a \Vdash \Diamond_i\Box_k p$. By our assumption we have $a \Vdash \Box_k\Diamond_l p$. Since $R_k ac$ we have $c \Vdash \Diamond_l p$. So there exists a d with $R_l cd$ such that $d \Vdash p$. Since $d \Vdash P$, by our valuation $R_j bd$. Thus we have confluence.

To show that (3) implies (1) suppose that $\mathcal{F}$ has (i, j, k, l) confluence. Let $\mathcal{M}$ be an arbitrary models based on $\mathcal{F}$ and $a \in W$ such that $a \Vdash \Diamond_i \Box_j p$. Then there exists a b with $R_i ab$ such that $b \Vdash \Box_j p$. To show that $a \Vdash \Box_k \Diamond_l p$, let $R_k ac$. Then applying confluence gives us a d with $R_j bd$ and $R_l cd$. Since $b \Vdash \Box_k p$ we have $d \Vdash p$. Then since $R_l cd$, $c \Vdash \Diamond_l p$. Since c was arbitrary, $a \Vdash \Box_k \Diamond_l p$. □

3.2. **First-order definability.** Note that each frame (W, R) can be viewed as a first-order structure for a language with signature $\{R_i\}_{i \in I}$. This motivates the following definition:

Definition 3.5. *A class of frames* K elementarily definable *if there is a first-order signature* $\mathcal{L}$ *and* $\mathcal{L}$*-formula* φ *such that for any* $\mathcal{L}$*-structure* M

$$M \vDash \varphi \iff M \in \mathsf{K}$$

Example 3.6. *Consider the class of frames* $\mathsf{K} = \{(W, R) \mid R \text{ is transitive}\}$. *We've seen that* $\Box p \to \Box\Box p$ *is valid in a frame* $\mathcal{F}$ *if and only if* $\mathcal{F} \in \mathsf{K}$. *So* K *is modally definable. But it is also definable by the first order formula:*

$$\forall x \forall y ((R(x, y) \wedge R(y, z)) \to R(x, z))$$

So in particular we have that for any frame (W, R),

$$(W, R) \Vdash \Box\varphi \to \Box\Box\varphi \iff (W, R) \vDash \forall x \forall y ((R(x, y) \wedge R(y, z)) \to R(x, z))$$

The correspondence results we have looked at thus far are all cases in which a modal formula defines an elementarily definable class of frames. But there are also examples of modally definable classes of frames that are not elementarily definable.

3.3. **Non-elementarily definable but modally definable classes of frames.**

Definition 3.7. *Let $\mathcal{F} = (W, R)$ be a frame. For $x, y \in W$, $i \in I$ and $n \in \mathbb{N}$ say that*

$$x \xrightarrow{R_{i_n}} y$$

if there is a sequence

$$x = x_0 \xrightarrow{R_i} \dots \xrightarrow{R_i} x_n = y$$

Lemma 3.8. *Let $\mathcal{F}$ be a frame and $w \in \mathcal{F}$. Let $\mathcal{M}$ be a model such that for any $x \in W$, $x \Vdash p$ if and only if for all $y \in W$, if $x \xrightarrow{R_n} y$, for some n, then $w \xrightarrow{R} y$.Then $w \Vdash \Box(\Box p \to p)$.*

Proof. Assume the hypothesis. Now let $w \xrightarrow{R} v$ be arbitrary such that $v \Vdash \Box p$. Let y be an arbitrary world with the property that $v \xrightarrow{R_n} y$. Now if $v = y$, then $w \xrightarrow{R} y$ since $w \xrightarrow{R} v$. Otherwise suppose that $v \neq y$. Then $v \xrightarrow{R} x_1 \xrightarrow{R_{n-1}} y$. Since $v \Vdash \Box p$, $x_1 \Vdash p$. So since $x_1 \xrightarrow{R_{n-1}} y$, it follows, by our valuation, that $w \xrightarrow{R} y$. Either way, $w \xrightarrow{R} y$. Hence $v \Vdash p$ and so $v \Vdash \Box p \to p$. Since v was arbitrary, $w \Vdash \Box(\Box p \to p)$. Thus the desired result holds. □

Lemma 3.9. *Suppose that $\mathcal{F} \Vdash \Box(\Box p \to p) \to \Box p$. Then R is transitive.*

Proof. Let $a \xrightarrow{R} b \xrightarrow{R} c$. Define a valuation as in the previous lemma. Then $a \Vdash \Box(\Box p \to p)$ by the above lemma. Since $\mathcal{F} \Vdash \Box(\Box p \to p) \to \Box p$, $a \Vdash \Box p$. So $b \Vdash p$. By our valuation $a \xrightarrow{R} c$. □

Definition 3.10. *Let $\mathcal{F} = (W, R)$ be a frame. We say that R is well-founded, or that $\mathcal{F}$ is a well-founded frame, if there is no infinite sequence*

$$w_0 \xrightarrow{R} w_1 \xrightarrow{R} w_2 \xrightarrow{R} \dots$$

Theorem 3.11. *The following are equivalent:*

(1) $\mathcal{F} \Vdash \Box(\Box p \to p) \to \Box p$.

(2) $\mathcal{F} \Vdash \Box(\Box \varphi \to \varphi) \to \Box \varphi$.

(3) $\mathcal{F}$ is a transitive and well-founded frame.

Proof. (1) is equivalent to (2) by substitution and trivial reasoning.

To show that (1) implies (3), suppose (1) holds. We have transitivity by our previous lemma. It remains to show well foundedness. For the sake of contradiction suppose that R is not well founded. Then there exists an infinite sequence

$$w_0 \xrightarrow{R} w_1 \xrightarrow{R} w_2 \xrightarrow{R} \dots$$

Define a valuation V such that $V(p) = W \setminus \{w_n \mid n \in \mathbb{N}\}$. Note that $w_1 \Vdash \neg p$. So $w_0 \Vdash \neg\Box p$. By (1), $w_0 \Vdash \neg\Box(\Box p \to p)$. This implies $w_0 \Vdash \Diamond\neg(\Box p \to p)$. So there is some v such that Rwv with $v \nVdash \Box p \to p$. This implies $v \Vdash \Box p$ and $b \nVdash p$. So since $b \nVdash p$, $b \in \{w_n \mid n \in \mathbb{N}\}$. That is, $v = w_n$ for some $n \in \mathbb{N}$. But since $v \Vdash \Box p$, and Rvw_{n+1}, it follows that $w_{n+1} \Vdash p$. That's a contradiction. So R must be well-founded.

To show that (3) implies (1) suppose that (3) holds. Let $\mathcal{M}$ be an arbitrary model based on $\mathcal{F}$ and $w \in W$ an arbitrary world such that $w \Vdash \Box(\Box p \to p)$. Let $w \xrightarrow{R} v_0$. For the sake of contradiction assume $v_0 \Vdash \neg\Box p$. Since $w \Vdash \Box(\Box p \to p)$, $v_0 \Vdash \Box p \to p$. So $v_0 \Vdash \neg\Box p$. So

$\Box$

This last example is significant since it shows that modal formulas can define classes of frames that are *not* first-order definable. In particular the class of transitive well-founded frames is not defined by any first-order formula. In the next example we will show that the class of frames on which the *McKinsey* formula is valid is also not first-order definable:

McKinsey formula: $\Box\Diamond p \to \Diamond\Box p$.

Example 3.12. *Let $F \subseteq \mathcal{P}(\mathbb{N})$ be a set of subsets of $\mathbb{N}$ that is closed under complements (i.e. if $X \in F$ then $\mathbb{N} \setminus X \in F$). Define a frame $\mathcal{F} = (W, R)$ by choosing an arbitrary object $w \notin \mathbb{N}$ and letting*

$$W = \{w\} \cup \mathbb{N} \cup (\mathbb{N} \times 2) \cup F$$

and defining R so that

$$w \xrightarrow{R} n \xrightarrow{R} (n,i) \xrightarrow{R} (n,i)$$

$$w \xrightarrow{R} X \xrightarrow{R} (n, \chi_X(n))$$

Here $\chi_X : X \to 2$ is the characteristic function of X and so maps each x to 1 if and only if $x \in X$.

Lemma 3.13. *If $F = \mathcal{P}(\mathbb{N})$ then $\mathcal{F} \Vdash \Box\Diamond p \to \Diamond\Box p$.*

Proof. Let $\mathcal{M}$ be a model based on $\mathcal{F}$. We want to show that for each $u \in W$,

$$u \Vdash \Box\Diamond p \to \Diamond\Box p$$

Start with the pairs (n,i) for $n \in \mathbb{N}$ and $i \in 2$. Since (n,i) sees itself, $(n,i) \Vdash \Box\Diamond\varphi$ only if $(n,i) \Vdash \Diamond\varphi$. Since (n,i) sees *only itself*, $(n,i) \Vdash \Diamond\varphi$ only if $(n,i) \Vdash \Box\varphi$. And again since (n,i) sees itself, $(n,i) \Vdash \Box\varphi$ only if $(n,i) \Vdash \Diamond\Box\varphi$. So we have that $(n,i) \Vdash \Box\Diamond\varphi$ only if $(n,i) \Vdash \Diamond\Box\varphi$. Thus $(n,i) \Vdash \Box\Diamond p \to \Diamond\Box p$.

Now let $u \in \mathbb{N} \cup F$ and suppose $u \Vdash \Box\Diamond\varphi$. Thus for any pair (n,i) such that $u \xrightarrow{R} (n,i)$, $(n,i) \Vdash \Diamond\varphi$. Since $(n,i) \Vdash \Diamond\varphi$ if and only $\Box\varphi$, we thus know that every pair (n,i) that u sees, $(n,i) \Vdash \Box\varphi$. Finally, since we know that u sees *some* pair, we have that $u \Vdash \Diamond\Box\varphi$. Hence $u \Vdash \Box\Diamond p \to \Diamond\Box p$.

It is only left to show that $w \Vdash \Box\Diamond p \to \Diamond\Box p$. So suppose that $w \Vdash \Box\Diamond\varphi$. Then for any $n \in \mathbb{N}$, $n \Vdash \Diamond\varphi$. Thus for any $n \in \mathbb{N}$, either $(n,0) \Vdash \varphi$, or $(n,1) \Vdash \varphi$. Let $X = \{n \mid (n,1) \Vdash \varphi\}$. Since $F = \mathcal{P}(\mathbb{N})$ it follows that $X \in W$ and $w \xrightarrow{R} X$. Now suppose that $X \xrightarrow{R} (n,i)$. If $n \in X$, then $i = 1$. Hence $(n,i) \Vdash \varphi$. If $n \notin X$ then $i = 0$. Since n is not in X, $(n,1) \nVdash \varphi$. But as we showed above, every n is such that *either* $(n,1) \Vdash \varphi$, or $(n,0) \Vdash \varphi$. Hence $(n,0) \Vdash \varphi$. So in either case, $(n,i) \Vdash \varphi$. So $X \Vdash \Box\varphi$. Since w sees X, $w \Vdash \Diamond\Box\varphi$. Thus we've shown that $w \Vdash \Box\Diamond p \to \Diamond\Box p$. Since w was the last point to show this completes the proof. □

Lemma 3.14. *If $F \neq \mathcal{P}(\mathbb{N})$ then $\mathcal{F} \nVdash \Box\Diamond p \to \Diamond\Box p$.*

Proof. Let $X \subseteq \mathbb{N}$ be such that $X \notin F$. Since F is closed under complements it follows that $\mathbb{N} \setminus X$ is also not in F. Define a valuation V such that

$$V(p) = \{(n,1) \mid n \in X\} \cup \{(n,0) \mid n \notin X\}$$

We'll first show that $w \Vdash \Box\Diamond p$. So let x be an arbitrary world that w sees. There are two cases: $x \in \mathbb{N}$ or $x \in F$. Suppose first that $x \in \mathbb{N}$. Then either $(x,1) \Vdash p$ or $(x,0) \Vdash p$. So $x \Vdash \Diamond p$. Now suppose that $x \in F$. Now if there is some $n \in X \cap x$ then $x \xrightarrow{R} (n,1)$ and $(n,1) \in V(p)$. Hence $(n,1) \Vdash p$. Hence $x \Vdash \Diamond p$. If there is no $n \in X \cap x$ then $x \subseteq \mathbb{N} \setminus X$. Note that since $x \in F$ it follows that $x \neq \mathbb{N} \setminus X$ and so there is some $n \in \mathbb{N}$ such that $n \notin x$ and $n \notin X$. So $x \xrightarrow{R} (n,0)$ and $(n,0) \in V(p)$. So $(n,0) \Vdash p$. Thus $x \Vdash \Diamond p$. In either case $x \Vdash \Diamond p$. So since x was chosen arbitrarily, $w \Vdash \Box\Diamond p$.

To complete the proof it thus suffices to show that $w \nVdash \Diamond\Box p$. So let x be arbitrary such that w sees x. There are two cases: $x \in \mathbb{N}$ and $x \in F$. Suppose first that $x \in \mathbb{N}$. Then since not both of $(x,0)$ and $(x,1)$ satisfy p, and x sees both of $(x,0)$ and $(x,1)$, it follows that $x \nVdash \Box p$. Now suppose that $x \in F$. If there is some $n \in x \setminus X$ then $x \xrightarrow{R} (n,1)$ and, $(n,1) \nVdash p$. Otherwise $x \setminus X = \emptyset$. So $x \subseteq X$. By assumption $X \notin F$ and $x \in F$, so there must be some $n \in X$ such that $n \notin x$. So $x \xrightarrow{R} (n,0)$ and $(n,0) \nVdash p$. In either case x sees something that does not satisfy p. So $x \nVdash \Box p$. In either case $x \nVdash \Box p$. Since x was chosen arbitrarily, $w \nVdash \Diamond\Box p$.

Thus we've shown that $w \nVdash \Box\Diamond p \to \Diamond\Box p$ and so if $F \neq \mathcal{P}(\mathbb{N})$, $\mathcal{F} \nVdash \Box\Diamond p \to \Diamond\Box p$. □

These two lemmas immediately imply the following theorem.

Theorem 3.15. $\mathcal{F} \Vdash \Box\Diamond p \to \Diamond\Box p$ *if and only if* $F = \mathcal{P}(\mathbb{N})$.

Corollary 3.16. *Let K be the class of frames defined by $\Box\Diamond p \to \Diamond\Box p$. Then K is not elementarily definable*

Proof. Suppose for reductio that K is defined by the first order formula φ. Let $\mathcal{F}$ be the frame defined above with $F = \mathcal{P}(\mathbb{N})$. By our theorem, $\mathcal{F} \in \mathsf{K}$ and so by asssumption $\mathcal{F} \vDash \varphi$. By the downward Lowenheim-Skolem Theorem there is some countable elementary substructure $\mathcal{F}'$ of $\mathcal{F}$. Since $\mathcal{F}'$ is an elementary substructure, its domain must contain w, each $n \in \mathbb{N}$ and each pair $(n, 1)$ and $(n, 0)$. But since it is countable, it cannot contain all of $\mathcal{P}(\mathbb{N})$. So by our theorem $\mathcal{F}' \notin \mathsf{K}$. □

3.4. **Generalizing confluence.** Let $F(\Phi, I)$ be a modal language with index set I. Let $n < \omega$ be a finite ordinal and $\sigma : n \to U$ a finite sequence $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ drawn from our index set I. We will write $w \xrightarrow{R_\sigma} v$ to mean

$$w = x_0 \xrightarrow{R_{\sigma_0}} x_1 \xrightarrow{R_{\sigma_1}} \ldots \xrightarrow{R_{\sigma_{n-1}}} x_n = v$$

Similarly, we write $\Box_\sigma \varphi$ to mean

$$\Box_{\sigma_0} \Box_{\sigma_1} \ldots \Box_{\sigma_{n-1}} \varphi$$

Lemma 3.17. *$w \Vdash \Box_\sigma \varphi$ if and only if for all v with $w \xrightarrow{R_\sigma} v$, $v \Vdash \varphi$.*

Proof.

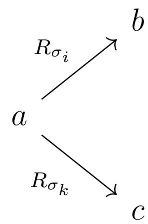
$$\begin{aligned} w \Vdash \Box_\sigma \varphi &\iff w \Vdash \Box_{\sigma_0} \Box_{\sigma_1} \ldots \Box_{\sigma_{n-1}} \varphi \\ &\iff (\forall x_1 \in R_{\sigma_0}(w))(\forall x_2 \in R_{\sigma_1}(x_1)) \ldots (\forall x_n \in R_{\sigma_{n-1}}(x_{n-1})) x_n \Vdash \varphi \\ &\iff (\forall v \in R_\sigma(w) v \Vdash \varphi \end{aligned}$$

□

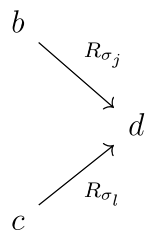
Lemma 3.18. *$w \Vdash \Diamond_\sigma \varphi$ if and only if there is some v with $w \xrightarrow{R_\sigma} v$ such that $v \Vdash \varphi$.*

Proof. Same as previous. □

Definition 3.19. *A frame has the* general confluence property *if and only if for any finite sequences* $\sigma_i, \sigma_j, \sigma_k$ *and* σ_l *drawn from* I *if there is a wedge*



there exists a $d \in W$ *such that*



Theorem 3.20. *The following are equivalent:*

(1) $\mathcal{F} \Vdash \Diamond_{\sigma_i}\Box_{\sigma_j} p \to \Box_{\sigma_k}\Diamond_{\sigma_l} p$

(2) $\mathcal{F} \Vdash \Diamond_{\sigma_i}\Box_{\sigma_j} \varphi \to \Box_{\sigma_k}\Diamond_{\sigma_l} \varphi$

(3) $\mathcal{F}$ *has the general confluence property.*

Proof. The proof is basically the same as the previous confluence proof. □

4. Completeness

4.1. **Basic soundness and completeness results.** In non-modal logic, we start with two notions of consequence:

(1) We say that $\Gamma \vdash \varphi$ if there is a syntactic proof of φ from the set Γ of sentences.

(2) We say that $\Gamma \vDash \varphi$ if every structure which makes Γ true also makes φ true.

We observe the following:

Theorem 4.1 (Completeness)**.** $\Gamma \vdash \varphi$ *if and only if* $\Gamma \vDash \varphi$.

Proof sketch. To show the left to right direction is just a matter of showing that the axioms are true in every structure and that the inference rules preserve truths in a structure. Going in the other direction, suppose that $\Gamma \vDash \varphi$ □

The goal of this section is to set up notions of proof and consequence for modal logic, and see to what extant they coincide.

Definition 4.2. *A formula φ is a* consequence *of a set of formulas Γ if and only if for any model $\mathcal{M}$, $\mathcal{M} \Vdash \Gamma$ implies $\mathcal{M} \Vdash \varphi$.*

In this section we will show that this notion of consequence can be captured syntactically. To start our discussion we'll look at the class of *normal modal logics*.

Definition 4.3. *A* normal modal logic *is a set L of formulas that contains*

- *All propositional tautologies;*
- $\Box(p \to q) \to (\Box p \to \Box q)$

and is closed under

- *Modus ponens: $\varphi \to \psi, \varphi \in L$ implies $\psi \in L$;*
- *Necessitation: $\varphi \in L$ implies $\Box\varphi \in L$.*
- *Substitution: $\varphi \in L$ implies $\varphi^* \in L$, where φ^* is a substitution instance of φ.*

Let L be a normal modal logic. We write $\vdash_L \varphi$ if $\varphi \in L$ and $\nvdash \varphi$ if $\varphi \notin L$. In this case we say that φ is a *theorem* of L, or an *L-theorem*.

Definition 4.4. *Let L be a normal modal logic. We say that φ is* deducible from Γ in L, *$\Gamma \vdash_L \varphi$ if there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that*

$$\vdash_L (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$$

Remark 4.5. *Let L and S be normal modal logics with $S \subseteq L$. Suppose that $\Gamma \subseteq \Gamma'$. Then $\Gamma \vdash_S \varphi$ implies $\Gamma' \vdash_L \varphi$.*

Lemma 4.6. *Let S be a normal modal logic and φ and ψ formulas. Then $\vdash_S \varphi \to \psi$ only if $\vdash_S \Box\varphi \to \Box\psi$.*

Proof.

$\vdash_S \varphi \to \psi$	Assumption
$\vdash_S \Box(\varphi \to \psi)$	by necessitation
$\vdash_S \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	by K and substitution
$\vdash_S \Box\varphi \to \Box\psi$	by MP

□

Corollary 4.7. *Let S be a normal modal logic. Then*

(1) $\vdash_S \varphi \leftrightarrow \psi$ only if $\vdash_S \Box\varphi \leftrightarrow \Box\psi$

(2) $\vdash_S \varphi \leftrightarrow \psi$ only if $\vdash_S \Diamond\varphi \leftrightarrow \Diamond\psi$.

Definition 4.8. *Let S and T be normal modal logics. We say that T* extends *S, or that S* refines *T, written $S \leq T$, if for all φ, $\vdash_S \varphi$ implies $\vdash_T \varphi$.*

Remark 4.9. *The relation $\leq$ partially orders the set of all normal modal logics. Let K be the logic that results from closing the set of propositional tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$ under modus ponens, necessitation and substitution. Obviously $K \subseteq L$ for any modal logic. Moreover the set of all formulas is also normal and contains every normal modal logic. Thus the partial order is bounded.*

Definition 4.10. *Let S be a normal modal logic and Γ a set of formulas. We say that Γ is* S-consistent *if $\Gamma \nvdash_S \bot$. We say that Γ is* maximally S-consistent *if*

(1) Γ is S-consistent and

(2) for any formula φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$

Lemma 4.11. *Let S be a normal modal logic and let Γ be S-consistent. Then for any formula φ either $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ is S-consistent.*

Proof. For the sake of contradiction suppose not. Then both $\Gamma \cup \{\varphi\} \vdash_S \bot$ and $\Gamma \cup \{\neg\varphi\} \vdash_S \bot$. This implies that there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ and $\psi_1, \ldots, \psi_m \in \Gamma$ such that

$$\vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n) \wedge \varphi \to \bot \text{ and } \vdash_S (\psi_1 \wedge \cdots \wedge \psi_m) \wedge \neg\varphi \to \bot$$

The by basic propositional logic we have

$$\vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \psi_1 \wedge \cdots \wedge \psi_m) \to \bot$$

Thus Γ is S-consistent. □

Lemma 4.12. *Let S be a normal modal logic and let Γ be S-consistent. Then there exists $\Gamma' \supseteq \Gamma$ such that Γ' is maximally S-consistent.*

Proof. Let $\varphi_1, \varphi_2, \ldots$ be a list of all formulas in our language. Applying the last lemma repeatedly we construct a sequence $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ such that

(1) Each Γ_i is S-consistent, and
(2) For each i, φ_i or $\neg\varphi_i$ is in Γ_i.

Then we let $\Gamma' = \bigcup_i \Gamma_i$.

Claim 4.13. *Γ' is maximally S-consistent.*

Proof. Maximality follows from construction. To show S-consistency assume that $\Gamma' \vdash_S \bot$. Then there exists $\varphi_1, \ldots, \varphi_n \in \Gamma'$ such that $\vdash_S \varphi_1 \wedge \cdots \wedge \varphi_n \to \bot$. Then there is some i such that $\varphi_1, \ldots, \varphi_n \in \Gamma_i$. This contradicts the fact that Γ_i is S-consistent. □

□

Definition 4.14. *Let S be a normal modal logic. The* canonical frame for S *is a frame*

$$\hat{\mathcal{S}} = (\hat{S}, R)$$

where $\hat{S}$ is the set of all maximally S-consistent sets of formulas and $X \xrightarrow{R_i} Y$ if and only if for any formula φ, if $\Box_i\varphi \in X$ then $\varphi \in Y$.

The canonical model for $\hat{\mathcal{S}}$ *is the model obtained by equipping* $\hat{\mathcal{S}}$ *with the valuation* V *that maps each variable* p *to the set of maximally* S*-consistent sets containing* p*. That is*

$$V(p) = \{X \in \hat{S} \mid p \in X\}$$

Lemma 4.15. *Let* X *be maximally* S*-consistent. Then*

(1) $\top \in X$, $\bot \in X$

(2) $\neg\varphi \in X$ *if and only if* $\varphi \notin X$

(3) $(\varphi \wedge \psi) \in X$ *if and only if* $\varphi \in X$ *and* $\psi \in X$

(4) $(\varphi \vee \psi) \in X$ *if and only if* $\varphi \in X$ *or* $\psi \in X$

(5) $(\varphi \to \psi) \in X$ *if and only if* $\varphi \notin X$ *or* $\psi \in X$

Proof. (1) $\vdash_S \neg\top$ which implies $\neg\top \notin X$ by S-consistency. So $\top \in X$ by maximality. Similarly, $\bot \notin X$ by S-consistency.

(2) If $\neg\varphi \in X$ then $\varphi \notin X$ by S-consistency. If $\neg\varphi \notin X$ then $\varphi \in X$ by maximality.

(3) Suppose that $\varphi \wedge \psi \in X$. If either φ or ψ is not in X then their negations in X by maximality. This contradicts consistency. Similarly if both φ and ψ are in X then by consistency $\neg(\varphi \wedge \psi)$ is not in X and so by maximality $\varphi \wedge \psi$ is in X.

(4) and (5) follow in a similar manner to (3). □

Lemma 4.16. $\Gamma \vdash_S \varphi$ *if and only if, for every maximally* S*-consistent set* X*, if* $\Gamma \subseteq X$ *then* $\varphi \in X$*.*

Proof. Suppose that $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq X$ for X maximally S-consistent. Then there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_S \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi$. Now since $\varphi_1, \ldots, \varphi_n \in \Gamma$, $\varphi_1, \ldots, \varphi_n \in X$, since $X \supseteq \Gamma$. By our last lemma $\varphi_1 \wedge \cdots \wedge \varphi_n \in X$. Since X is maximally S-consistent, $\varphi \in X$.

Conversely, suppose that for all $X \in \hat{S}$, $\Gamma \subseteq X$ only if $\varphi \in X$. Then there is no $X \in \hat{S}$ with $\Gamma \cup \{\neg\varphi\} \subseteq X$. So $\Gamma \cup \{\neg\varphi\}$ is not S-consistent. Thus $\Gamma \vdash_S \varphi$. □

Lemma 4.17. *Let $\hat{\mathcal{S}} = (\hat{S}, R)$ be the canonical frame for S and let $X \in \hat{S}$. Then for any formula φ, $\Box\varphi \in X$ if and only if for all $Y \in \hat{S}$ such that $X \xrightarrow{R} Y$, $\varphi \in Y$.*

Proof. The left-to-right direction follows immediately from the definition of R. To show the other direction suppose that for all for all $Y \in \hat{S}$ such that $X \xrightarrow{R} Y$, $\varphi \in Y$. Let $N = \{\psi \mid \Box\psi \in X\}$. Then for any $Y \in \hat{S}$, if $N \subseteq Y$ then $X \xrightarrow{R} Y$. So by our assumption, $\varphi \in Y$. By our last lemma $N \vdash_S \varphi$. So we can find $\varphi_1, \ldots, \varphi_n \in N$ such that $\vdash_S \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi$. By a previous lemma $\vdash \Box(\varphi_1 \wedge \cdots \wedge \varphi_n \to \Box\varphi)$. Thus $\vdash \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \to \Box\varphi$. Thus $X \vdash_S \Box\varphi$ and further $\Box\varphi \in X$ since X is maximally S-consistent. $\Box$

Theorem 4.18. *Let $(\hat{\mathcal{S}}, V)$ be the canonical model for S. Then for any $X \in \hat{S}$ and formula φ,*

$$X \Vdash \varphi \iff \varphi \in X$$

Proof. We proceed by formula induction. By a previous lemma, if $X \Vdash \top$ then $\top \in X$ and if $X \nVdash \bot$ then $\bot \notin X$. This takes care of the constants.

For the propositional variables we have

$$\begin{aligned} X \Vdash p &\iff x \in V(p) \\ &\iff p \in X \end{aligned}$$

The first equivalence is by definition of $\Vdash$ and the second by definition of V.

Now suppose it holds for φ and ψ. Then

$$\begin{aligned} X \Vdash \varphi \to \psi &\iff X \nVdash \varphi \text{ or } X \Vdash \psi \\ &\iff \varphi \notin X \text{ or } \psi \in X \\ &\iff (\varphi \to \psi) \in X \end{aligned}$$

The first equivalence is by definition of $\Vdash$, the second is by induction, and the third is by maximal S-consistency.

Finally suppose it holds for φ.

$$\begin{aligned} X \Vdash \Box\varphi &\iff \forall Y < X \; X \Vdash \varphi \\ &\iff \forall Y < X \; \varphi \in Y \\ &\iff \Box\varphi \in Y \end{aligned}$$

The first is by definition, second by induction, and third by our previous lemma. □

Corollary 4.19. *Let $(\hat{\mathcal{S}}, V)$ be the canonical model for S. Then $(\hat{\mathcal{S}}, V) \Vdash S$.*

Proof. Let $X \in \hat{S}$ be arbitrary. By our previous lemma $X \Vdash S$ if and only if $S \subseteq X$. And since X is maximally S-consistent, $S \subseteq X$. □

Corollary 4.20. *$(\hat{\mathcal{S}}, V) \Vdash \varphi$ if and only if $\vdash_S \varphi$.*

Proof.

$$\begin{aligned} (\hat{\mathcal{S}}, V) \Vdash \varphi &\iff \forall X \in \hat{S}, X \Vdash \varphi \\ &\iff \forall X \in \hat{S}, \varphi \in X \\ &\iff \emptyset \vdash_S \varphi \\ &\iff \vdash_S \varphi \end{aligned}$$

□

Theorem 4.21. *Let S be a normal modal logic. Then the following are equivalent.*

(1) $(\hat{\mathcal{S}}, V) \Vdash \varphi$

(2) $\vdash_S \varphi$

(3) $S \vDash \varphi$

Proof. (1) implies (2) by the last corollary. (2) implies (3) by soundness. And (3) implies (1) by a previous corollary. □

It may be good to state this in English so it sticks in your mind. How should we think of the canonical model for a normal modal logic S? Well it is that model such that the formulas true on it are *precisely* the theorems of S.

Corollary 4.22. *Let S be a normal modal logic and Γ a collection of formulas. Let*

$$\Gamma^* = \{\Box^*\varphi \mid \varphi \in \Gamma\}$$

where $\Box^*\varphi$ is the result of prefixing any number of $\Box$'s to φ. Define a model $\hat{\mathcal{S}}_\Gamma$ be letting $\hat{S}_T = \{X \in \hat{S} \mid \Gamma^* \subseteq X\}$ and defining R and V as in the canonical model. Then the following are equivalent:

(1) $(\hat{\mathcal{S}}_\Gamma, V) \Vdash \varphi$.

(2) $\Gamma \vdash_S \varphi$.

(3) $\Gamma \vDash_S \varphi$.

Proof. Follows exactly as before. $\Box$

The $(\hat{\mathcal{S}}_\Gamma, V)$ is thus that model that makes true precisely those formulas that are S-deducible from Γ. Note in particular that we have

Theorem 4.23 (Completeness)**.** $\Gamma \vdash_S \varphi$ *if and only if* $\Gamma \vDash_S \varphi$

4.2. **Kripke completeness.** The last section showed that the consequences of a theory in a normal modal logic can be captured syntactically. But there is another more interesting sense of completeness that we can study in modal logic. Let S be a normal modal logic. We will write $\mathrm{Mod}(S)$ for the class of all frames $\mathcal{F}$ on which every member of S is valid. For a class of K of frames, we write $\mathrm{Log}(\mathsf{K})$ for the set of all formulas valid on every frame in K. Now for a given logic S and class of frames K, we say that S is *sound* with respect to K if and only if $S \subseteq \mathrm{Log}(\mathsf{K})$. We say that S is *complete* with respect to K if $\mathrm{Log}(\mathsf{K}) \subseteq S$. Finally, we say that S is *Kripke complete* if and only if $S = \mathrm{Log}(\mathrm{Mod}(S))$, i.e. if S is sound and complete with respect to the class of all of its models.

Our motivating question: which normal modal logics are Kripke complete? Note that S is Kripke complete if and only if it is sound and complete with respect to *some* class of frames. If S is Kripke complete, then obviously it is sound and complete with respect to $\mathrm{Mod}(S)$. Suppose that S is sound and complete with respect to K. Then we have $\mathsf{K} \subseteq \mathrm{Mod}(S)$ and so $\mathrm{Log}(\mathrm{Mod}(S) \subseteq \mathrm{Log}(\mathsf{K})$. So since $\mathrm{Log}(\mathsf{K}) \subseteq S$ by assumption, it follows that $\mathrm{Log}(\mathrm{Mod}(S) \subseteq S$. Trivially $S \subseteq \mathrm{Log}(\mathrm{Mod}(S))$ and so S is Kripke complete just in case it is sound and complete with respect to some class of frames.

We can easily prove a sufficient condition on Kripke completeness that will guide our study into this question in what follows.

Definition 4.24 (Canonicity)**.** *A normal modal logic is* canonical *if it is valid on its canonical frame.*

Recall that each normal modal logic is true on its canonical model. Being canonical is a much stronger property: it must be true not only on its canonical model, but true on any model based on its canonical frame.

Theorem 4.25. *Let S be canonical and let $(\hat{\mathcal{S}}, V)$ be its canonical model. Then the following are equivalent:*

(1) $\hat{\mathcal{S}} \Vdash \varphi$.

(2) $(\hat{\mathcal{S}}, V) \Vdash \varphi$.

(3) $\vdash_S \varphi$.

(4) $\models_S \varphi$.

(5) $\varphi \in \mathrm{Log}(\mathrm{Mod}(S))$.

Proof. The implication from (1) to (2) follows by definition. The implication from (2) to (3) and from (3) to (4) follow by the previous theorem and soundness. The implication from (4) to (5) follows by definition. Finally to complete the circle, the implication from (5) to (1) follows from the assumption that S is canonical. □

Corollary 4.26. *Let S be canonical. Then S is Kripke complete.*

4.3. **Canonical modal logics.** If a normal modal logic is canonical, it is Kripke complete. But which normal modal logics are canonical? The goal of this section is to explore that question. Our first method at exploring it will be to use correspondence theory. Note first that the basic modal logic K is canonical trivially since it is valid on every frame. To show that, for instance T, the normal modal logic axiomatized by $\Box p \to p$, is canonical, it suffices to show that the accessibility relation in T's canonical frame is *reflexive*. Let's start there.

Proposition 4.27. *Let S be axiomatized by $\Box p \to p$. Then $\hat{\mathcal{S}} \Vdash \Box p \to p$.*

Proof. By correspondence theory it suffices to show that the accessibility relation R is reflexive on $\hat{S}$. So let $X \in \hat{S}$ be arbitrary and suppose that $\Box\varphi \in X$ for an arbitrary formula φ. By assumption $\Box p \to p \in S$ and S is closed under substitution. So $\Box\varphi \to \varphi \in S$. Now since X is maximally S-consistent, $S \subseteq X$. So $\Box\varphi \to \varphi \in X$. By maximal S-consistency, $\varphi \in X$. Therefore, by the definition of R, $X \xrightarrow{R} X$. □

Proposition 4.28. *Let S be axiomatized by $p \to \Box\Diamond p$. Then $\hat{\mathcal{S}} \Vdash p \to \Box\Diamond p$.*

Proof. By correspondence theory it suffices to show that R is symmetric on $\hat{S}$. So suppose that $X \xrightarrow{R} Y$ and $\Box\varphi \in Y$. Then by an old lemma $\Diamond\Box\varphi \in X$. We know that $p \to \Box\Diamond p \in X$. So by maximal S-consistency, $\Diamond\Box\varphi \to \varphi \in X$. So by maximal S-consistency $\varphi \in X$. Therefore $Y \xrightarrow{R} X$ and so R is symmetric. □

Proposition 4.29. *Let S be axiomatized by $\Box p \to \Box\Box p$. Then $\hat{\mathcal{S}} \Vdash \Box p \to \Box\Box p$.*

Proof. By correspondence theory, it suffices to show that R is transitive. Suppose that

$$X \xrightarrow{R} Y \xrightarrow{R} Z$$

and let $\Box\varphi \in X$. Then, since $\Box\varphi \to \Box\Box\varphi \in X$, we have $\Box\Box\varphi \in X$. So since $X \xrightarrow{R} Y$, $\Box\varphi \in Y$. And since $Y \xrightarrow{R} Z$, $\varphi \in Z$. Thus $X \xrightarrow{R} Z$ and so R is transitive. □

Corollary 4.30. *The following are canonical (and thus Kripke complete): $K, KT, K4, K5, S4, S5, KB$.*

To get a more general result, we'll now show that any logic that is axiomatized by the confluence axioms is canonical and so Kripke complete.

Lemma 4.31. *Let S be a normal modal logic and $X, Y \in \hat{S}$. Let σ be a sequence of labels. Suppose that for any φ, $\Box_\sigma \varphi \in X$ implies $\varphi \in Y$. Then there exists as sequence*

$$X = X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \dots \xrightarrow{i_n} X_n = Y$$

Proof. Suppose not. Let σ be the shortest sequence of labels for which the result fails.

Case 1: σ has length 0. Then $\varphi \in X$ implies $\varphi \in Y$. So $X = Y$ since X and Y are maximal.

Case 2: : σ has length $n+1$ for some n. Let $\tau = \sigma \setminus i_{n+1}$. Let

$$\Gamma = \{\varphi \mid \Box_\tau \varphi \in X\} \cup \{\Diamond_{i_{n+1}} \varphi \mid \varphi \in Y\}$$

We want to show that Γ is S-consistent so that we can use a previous lemma to extend to a maximally S-consistent set.

Claim 4.32. *Γ is maximally S-consistent.*

Proof of claim. By way of contradiction suppose that Γ is not S-consistent. Then there exists $\varphi_1, \dots, \varphi_n \in X$ and $\Diamond_{i_{n+1}} \psi_1, \dots, \Diamond_{i_{n+1}} \psi_m \in Y$ such that

$$\vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \Diamond_{i_{n+1}} \psi_1 \wedge \cdots \wedge \Diamond_{i_{n+1}} \psi_m) \to \bot$$

$$\Longrightarrow \vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \neg(\Diamond_{i_{n+1}} \psi_1 \wedge \cdots \wedge \Diamond_{i_{n+1}} \psi_m))$$

$$\Longrightarrow \vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \Box_{i_{n+1}} \neg(\psi_1 \wedge \cdots \wedge \psi_m)$$

$$\Longrightarrow \vdash_S \Box_\tau(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \Box_\sigma \neg(\psi_1 \wedge \cdots \wedge \psi_m)$$

(The last implication follows from a repeated application of a previous lemma (essentially it follows from repeated application of necessitation, modus ponens and the K axiom.)

Note that for each φ_i, $\Box_\tau \varphi_i \in X$. Since X is maximally S-consistent, $\Box_\tau(\varphi_1 \wedge \cdots \wedge \varphi_n) \in X$, which implies that $\Box_\sigma \neg(\psi_1 \wedge \cdots \wedge \psi_m) \in X$. So we have $\neg(\psi_1 \wedge \cdots \wedge \psi_m) \in Y$, which contradicts the fact that $\psi_i \in Y$ $\Box$

So let X' be a maximally S-consistent set extending Γ, which we know to exist by the a previous result. Recall that our assumption is that σ has lenght $n+1$ and that, moreover, σ is the shortest sequence for which our result fails. By construction, if $\Box_\tau \in X$, then $\varphi \in X'$. Since σ was the shortest sequence for which the result fails, we thus have

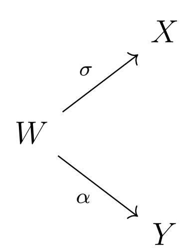
$$X = X_0 \xrightarrow{i_1} \ldots \xrightarrow{i_n} X_n = X'$$

Furthermore, again by construction, we have that $\varphi \in Y$ only if $\Diamond_{i_{n+1}}\varphi \in X'$. Thus by the definition of the accessibility relation in the canonical frame for S, $X' \xrightarrow{i_{n+1}} Y$. This is a contradiction.

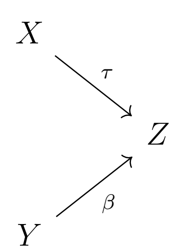
$\Box$

Theorem 4.33. *Let σ, τ, α and β be sequences of labels and suppose that $\Diamond_\sigma \Box_\tau p \to \Box_\alpha \Diamond_\beta p \in S$. Then $\hat{\mathcal{S}} \Vdash \Diamond_\sigma \Box_\tau p \to \Box_\alpha \Diamond_\beta p$*

Proof. By correspondence theory, it suffices to show that for any wedge



There exists some $Z \in \hat{S}$ such that



Let $\Gamma = \{\varphi \mid \Box_\tau \in X\} \cup \{\psi \mid \Box_\beta \in Y\}$

Claim 4.34. Γ *is* S*-consistent.*

Proof of claim. Suppose not. Then there are $\Box_\tau\varphi_1, \ldots, \Box_\tau\varphi_n \in X$ and $\Box_\beta\psi_1, \ldots, \Box_\beta\psi_m \in Y$ such that

$$\vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \psi_1 \wedge \cdots \wedge \psi_m) \to \bot$$
$$\Longrightarrow \vdash_S (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \neg(\psi_1 \wedge \cdots \wedge \psi_m)$$
$$\Longrightarrow \vdash_S \Box_\tau(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \Box_\tau\neg(\psi_1 \wedge \cdots \wedge \psi_m)$$

Since $\Box_\tau\varphi_1, \ldots, \Box_\tau\varphi_n \in X$ it follows from maximal consistency that $\Box_\tau(\varphi_1 \wedge \cdots \wedge \varphi_n) \in X$. Thus $\Box_\tau\neg(\psi_1 \wedge \cdots \wedge \psi_m) \in X$. Since $W \xrightarrow{\sigma} X$, $\Diamond_\sigma\Box_\tau\neg(\psi_1 \wedge \cdots \wedge \psi_m) \in W$. By maximal S-consistency, $\Box_\alpha\Diamond_\beta\neg(\psi_1 \wedge \cdots \wedge \psi_m) \in W$. Finally, since $W \xrightarrow{\alpha} Y$, $\Diamond_\beta\neg(\psi_1 \wedge \cdots \wedge \psi_m) \in Y$, contradicting the fact that $\Diamond_\beta\psi_i \in Y$ for each $1 \leq i \leq m$. $\Box$

Since Γ is S-consistent, we can extend it to a maximally S-consistent set Z. By construction of Γ we then have $X \xrightarrow{\tau} Z$ and $Y \xrightarrow{\beta} Z$. $\Box$

Corollary 4.35. *Let* S *be axiomatized by confluence axioms. Then* S *is canonical and so Kripke complete.*

This finishes, at least for the time being, our discussion of completeness. We introduced a general sufficient condition on being Kripke complete, and proved a variety of normal modal logics were Kripke complete via that sufficient condition. Going forward, we are going to start introducing various types of maps between frames and models using these to prove some general results about them.

5. Filtrations and Bounded Morphisms

At this point we have introduce frames and models and proved some results about the relationship between normal modal logics and these objects. In mathematics, objects are not normally studied in isolation, however. Rather, we study both those objects and various *mappings* between them. In this section, we'll introduce what some of these mappings are. For the time being our interests will remain more model theoretic. Perhaps later we will shift to a more algebraic or category theoretic view on modal logic.

Definition 5.1. *Let $\mathcal{F} = (W, R)$ and $\mathcal{H} = (W', R')$ be frames. A* morphism

$$f : \mathcal{F} \to \mathcal{H}$$

is a function $f : W \to W'$ that is such that

$$w \xrightarrow{R_i} v \implies f(w) \xrightarrow{R_i'} f(v)$$

If $\mathcal{M} = (\mathcal{F}, V)$ and $\mathcal{N} = (\mathcal{H}, V')$ are models a morphism

$$f : \mathcal{M} \to \mathcal{N}$$

is a morphism $f : \mathcal{F} \to \mathcal{H}$ with the property that

$$w \in V(p) \implies f(w) \in V'(p)$$

Definition 5.2. *Let $\mathcal{F}$ and $\mathcal{H}$ be frames. A* bounded morphism

$$f : \mathcal{F} \to \mathcal{H}$$

is a morphism from $\mathcal{F}$ to $\mathcal{H}$ that is such that, if

$$f(w) \xrightarrow{R_i'} v$$

then there exists $w' \in W$ such that

$$w \xrightarrow{R_i} w' \text{ and } f(w') = v$$

A bounded morphism $f : \mathcal{M} \to \mathcal{N}$ between models is a morphism from $\mathcal{M}$ to $\mathcal{N}$ that is also a bounded morphism between their underlying frames.

Remark 5.3. *Let $f : \mathcal{F} \to \mathcal{H}$ be a morphism. For each accesibility relation relation R_i, let $R_i(w) = \{v \mid R_i wv\}$. For each $X \subseteq W$ let $f(X) = \{w' \in W' \mid f(w) = w'$ for some $w \in W\}$. Then f is bounded if and only if*

$$R_i'(f(w)) \subseteq f(R_i(w))$$

Morever by the definition of morphism, we have

$$f(R_i(w)) \subseteq (R_i(f(w)))$$

Thus with bounded morphisms we have

$$f(R_i(w)) = R_i(f(w))$$

Theorem 5.4. *Let $f : \mathcal{M} \to \mathcal{N}$ be a bounded morphism. Then for all $w \in W$*

$$w \Vdash \varphi \iff f(w) \Vdash \varphi$$

Proof. If $w \Vdash \top$ then $f(w) \Vdash \top$. Also, if $w \nVdash \bot$ then $w \nVdash \bot$. For the propositional variables

$$\begin{aligned} w \Vdash p &\iff w \in V(p) \\ &\iff f(w) \in V'(p) \\ &\iff f(w) \Vdash p \end{aligned}$$

For the conditional we have

$$\begin{aligned} w \Vdash \varphi \to \psi &\iff w \nVdash \varphi \text{ or } w \Vdash \psi \\ &\iff f(w) \nVdash \varphi \text{ or } f(w) \Vdash \psi \\ &\iff f(w) \Vdash \varphi \to \psi \end{aligned}$$

Suppose that $w \Vdash \Box_i\varphi$ and let $v \in R_i'(f(w))$. Then $v \in f(R(w))$. Let $w' \in R(w)$ be such that $f(w') = v$. Then since $w \Vdash \Box_i\varphi$, $w' \Vdash \varphi$. By the induction hypothesis, $v \Vdash \varphi$. Since $v \in R_i'(f(w))$ was chosen arbitrarily, $f(w) \Vdash \Box_i\varphi$.

Conversely, suppose that $f(w) \Vdash \Box_i\varphi$ and let $w' \in R_i(w)$. Then since f is a morphism, $f(w') \in R(f(w))$. Thus $f(w') \Vdash \varphi$. By the induction hypothesis, $w' \Vdash \varphi$. Since w' was arbitrary, $w \Vdash \Box_i\varphi$. $\Box$

Corollary 5.5. *Let $f : \mathcal{M} \to \mathcal{N}$ be a surjective bounded morphism. Then*

$$\mathcal{M} \Vdash \varphi \iff \mathcal{N} \Vdash \varphi$$

Corollary 5.6. *Let $f : \mathcal{F} \to \mathcal{H}$ be bounded morphism between frames. Then if f is surjective, $\mathcal{F} \Vdash \varphi$ only if $\mathcal{H} \Vdash \varphi$.*

Proof. Suppose that $\mathcal{F} \Vdash \varphi$. Let $\mathcal{N}$ be an arbitrary model based on $\mathcal{H}$. Define a valuation V on $\mathcal{F}$ by $w \in V(p)$ if and only if $\mathcal{M}, f(w) \Vdash p$. Then where $\mathcal{M} = (\mathcal{F}, V)$, the map $f : \mathcal{M} \to \mathcal{N}$ is a bounded morphism. Since $\mathcal{F} \Vdash \varphi$, $\mathcal{M} \Vdash \varphi$. By our last corollary $\mathcal{N} \Vdash \varphi$. $\Box$

5.1. **Anti-correspondence results?** Suppose you wanted to show that a given structural condition was *not* captured by any modal formula. Here is one way to do that. Find a frame $\mathcal{F}$ that has that structural condition. Then construct a surjetive, bounded morphism $f : \mathcal{F} \to \mathcal{H}$ to a frame $\mathcal{H}$ that lacks that condition. By our last corollary, $\mathcal{F} \Vdash \varphi$ only if $\mathcal{H} \Vdash \varphi$. Thus no formula captures the structural condition in question.

Example 5.7. *Let $\mathcal{F} = (W, R)$ be the frame defined by*

- $W = \{0, 1\}$
- $R = \{(0, 1), (1, 0)\}$

Let $\mathcal{H} = (\{\star\}, \{(\star, \star)\}$ be the smallest reflexive frame. Let $f : W \to \{\star\}$ be a function. Then $f : \mathcal{F} \to \mathcal{H}$ is a bounded, surjective morphism. Obviously f is a surjective map. Since $\star$ sees $\star$, for all $w, v \in W$, if Rwv then $f(w)$ sees $f(v)$, since $f(w) = f(v) = \star$. Now suppose *that $f(w)$ sees v in $\mathcal{H}$. We have two cases. First $w = 0$. Then then since 0 sees 1 and $f(1) = \star = v$ we are done. Otherwise $w = 1$. Since 1 sees 0 and $f(0) = \star = v$, we are done.*

So f is a surjective, bounded morphism. By our last corollary, $\mathcal{F} \Vdash \varphi$ only if $\mathcal{H} \Vdash \varphi$. But notice that $\mathcal{F}$ is anti-reflexive and $\mathcal{H}$ is reflexive, and so not anti-reflexive. The result is that no modal formula defines the class of anti-reflexive frames.

5.2. **Filtration.**

Definition 5.8. *A set Γ is* closed under subformulas *if for all formulas φ and ψ*

(i) $\neg\varphi \in \Gamma$ *only if* $\varphi \in \Gamma$

(ii) $(\varphi \to \psi) \in \Gamma$ *only if* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.

(iii) $\Box_i\varphi \in \Gamma$ *only if* $\varphi \in \Gamma$

Definition 5.9. *Let Γ be a set of formulas closed under subformulas. A* Γ-filtration *between $(\mathcal{F}, V)$ and $(\mathcal{H}, V')$ is a surjective morphism*

$$f : \mathcal{F} \to \mathcal{H}$$

such that

(i) $V' = f \circ V$

(ii) If $\Box\varphi \in \Gamma$ *and* $f(v) \in R(f(w))$ *then* $w \Vdash \Box\varphi$ *only if* $v \Vdash \varphi$.

Proposition 5.10. *Let $f : \mathcal{M} \to \mathcal{N}$ be bounded surjective morphism. Then f is a Γ-filtration for any set Γ of formulas closed under subformulas.*

Proof. Trivially f is a surjective morphism such that $V' = f \circ V$. So suppose that $f(v) \in R'(f(w))$ and $w \Vdash \Box\varphi$. By our theorem for surjective bounded morphisms between models it follows that $f(w) \Vdash \Box\varphi$. This implies that $f(v) \Vdash \varphi$. Hence $v \Vdash \varphi$ since f is a surjective bounded morphism. □

Theorem 5.11. *Let $f : \mathcal{M} \to \mathcal{N}$ be a Γ-filtration. Then for every $\varphi \in \Gamma$,*

$$w \Vdash \varphi \iff f(w) \Vdash \varphi$$

Proof. The proof is by induction. We will only show the $\Box$ step. Let $w \Vdash \varphi$ and let $v \in Rf(w)$ be arbitrary. Since f is surjective, there is some $w' \in W$ such that $f(w') = v$. By condition (ii) of Γ-filtration, $w' \Vdash \varphi$. By induction, $f(w) = v \Vdash \varphi$. Since v was arbitrary, $f(w) \Vdash \Box\varphi$. Conversely, suppose that $f(w) \Vdash \Box\varphi$ and let $v \in R(w)$. Thus $f(v) \in Rf(w)$. Thus $f(v) \Vdash \varphi$. But induction, $v \Vdash \varphi$. Since v was arbitrary, $w \Vdash \Box\varphi$. $\Box$

Example 5.12 (Left-filtration)**.** *Let $\mathcal{M} = (W, R, V)$ be a model and Γ a set of formulas closed under subformulas. Define an equivalence relation $\sim$ on W by $w \sim v$ if w and v agree on Γ in $\mathcal{M}$ (i.e. $w \Vdash \varphi$ iff $v \Vdash \varphi$ for all $\varphi \in \Gamma$). We define the* left Γ-filtration $^{\Gamma}\mathcal{M} = (^{\Gamma}W, {}^{\Gamma}R, {}^{\Gamma}V)$ *of $\mathcal{M}$ as follows:*

- $^{\Gamma}W = W/\sim = \{[w] \mid w \in W\}$ *(where $[w] = \{v \in W \mid w \sim v\}$.*
- $[w] \xrightarrow{^{\Gamma}R_i} [v]$ *if and only if there exist $w' \in [w]$ and $v' \in [v]$ such that $w' \xrightarrow{R_i} v'$.*
- $[w] \in {}^{\Gamma}V(p)$ *if and only if some $w' \in [w]$ is such that $w \in V(p)$*

Theorem 5.13. *For any model $\mathcal{M}$ and subformula closed set Γ, the canonical map*

$$[\cdot] = w \mapsto [w] : \mathcal{M} \to {}^{\Gamma}\mathcal{M}$$

is a Γ-filtration.

Proof. (1) Clearly, $[\cdot]$ is surjective (since the projection maps onto quotient spaces are always surjective). If w sees v in $\mathcal{M}$. Then $[w]$ sees $[v]$ in $^{\Gamma}\mathcal{M}$ since $w \in [w]$ and $v \in [v]$.

(2) For the valuation, let $p \in \Gamma$. If $w \in V(p)$ then $[w] \in {}^{\Gamma}V(p)$ by definition. Suppose that $[w] \in {}^{\Gamma}V(p)$. Then there exists $w' \in [w]$ such that $w' \in V(p)$. Since $p \in \Gamma$, $w' \Vdash p$ if and only if $w \Vdash p$. So $w \in V(p)$.

(3) Let $\Box\varphi \in \Gamma$. Let $w, v \in W$ be such that $[w]$ sees $[v]$. Suppose that that $w \Vdash \Box\varphi$. Then since $[w]$ sees $[v]$, there exists $w' \in [w]$ and $v' \in [v]$ such that w' sees v'. Then

we have

$$\begin{aligned} w \Vdash \Box\varphi &\implies w' \Vdash \Box\varphi && \text{since } w \sim w' \\ &\implies v' \Vdash \varphi && \text{since } w' \text{ sees } v' \\ &\implies v \Vdash \varphi && \text{since } v' \sim v \end{aligned}$$

□

Example 5.14 (Right-filtration)**.** *Let $\mathcal{M} = (W, V, R)$ be a model and Γ a set of formulas closed under subformulas. We define the* right Γ-filtration $\mathcal{M}^\Gamma = (W^\Gamma, R^\Gamma, V^\Gamma)$ *of $\mathcal{M}$ as follows*

- $W^\Gamma = W/\sim$
- $[w] \xrightarrow{R_i^\Gamma} [v]$ *if and only if for all $\Box\varphi \in \Gamma$, $w' \in [w]$ and $v' \in [v]$, if $w' \Vdash \Box\varphi$ then $v' \Vdash \varphi$.*
- $[w] \in V^\Gamma(p)$ *if and only if $w' \in V(p)$, for all $w' \in [w]$.*

Theorem 5.15. *For any model $\mathcal{M}$ and subformula closed set Γ, the canonical map*

$$[\cdot] : \mathcal{M} \to \mathcal{M}^\Gamma$$

is a Γ-filtration.

Proof. (1) $[\cdot]$ is trivially a surjection. To show that it is a morphism, let w see v. Let $\Box\varphi \in \Gamma$, $w' \in [w]$ and $v' \in [v]$ and suppose that $w' \Vdash \Box\varphi$. Then since $w \sim w'$, $w \Vdash \Box\varphi$. Since w sees v, $v \Vdash \varphi$. Finally since $v \sim v'$, $v' \Vdash \varphi$.

(2) By definition $[w] \in V^\Gamma(p)$ only if $w \in V(p)$. Conversely suppose that $w \in V(p)$ and let $w' \in [w]$ be arbitrary. Since $p \in \Gamma$ and $w \sim w'$ $w' \Vdash p$ and so $w' \in V(p)$. Since w' was arbitrary, $[w] \in V^\Gamma(p)$.

(3) Let $\Box\varphi \in \Gamma$. Let $w, v \in W$ be such that $[w]$ sees $[v]$. The definition of the accessibility relation in $\mathcal{M}^\Gamma$ then ensures that $w \Vdash \Box\varphi$ only if $w' \Vdash \varphi$.

□

Definition 5.16. *Let $\mathcal{M} = (W, R, V)$ and $\mathcal{N} = (W, R', V')$ be two models with the same underlying set of worlds W. We write $\mathcal{M} \leq_\Gamma \mathcal{N}$ if and only if*

(1) $R(w, v)$ implies $R'(w, v)$.

(2) $w \in V(p)$ implies $w \in V'(p)$, for $p \in \Gamma$

Theorem 5.17. *Let $\mathcal{M} = (W, R, V)$ be a model and Γ a subformula closed set. Let $\mathcal{N} = (W/\sim, R', V')$ be any model whose underlying domain is $W/\sim$. Then the canonical map*

$$[\cdot] : \mathcal{M} \to \mathcal{N}$$

is a Γ-filtration if and only if

$$^\Gamma\mathcal{M} \leq_\Gamma \mathcal{N} \leq_\Gamma \mathcal{M}^\Gamma$$

Proof. Suppose first that $[\cdot]$ is a Γ filtration.

(1) Let $\Box\varphi \in \Gamma$. Then

$$
\begin{aligned}
[w] \xrightarrow{^\Gamma R_i} [v] &\implies (\exists w' \in [w])(\exists v' \in [v]) w \xrightarrow{R_i} v && \text{by the definition of } {}^\Gamma R_i \\
&\implies [w] \xrightarrow{R'_i} [v] && \text{since } [\cdot] : \mathcal{M} \to \mathcal{N} \text{ is a morphism} \\
&\implies (\forall w' \in [w])(\forall v' \in [v])(w' \Vdash \Box\varphi \implies v' \Vdash \varphi) && \text{since } f \text{ is a filtration} \\
&\implies [w] \xrightarrow{R_i^\Gamma} [v] && \text{by the definition of } R^\Gamma
\end{aligned}
$$

(2) Let $p \in \Gamma$. Then

$$
\begin{aligned}
b \in {}^\Gamma V(p) &\implies (\exists w' \in [w]) w' \in V(p) && \text{by definition of } {}^\Gamma V \\
&\implies [w] \in V'(p) && \text{since } [\cdot] \text{ is a filtration} \\
&\implies (\forall w' \in [w]) w' \in V(p) && \text{since } p \in \Gamma \\
&\implies [w] \in V^\Gamma(p)
\end{aligned}
$$

Conversely suppose that $^\Gamma\mathcal{M} \leq_\Gamma \mathcal{N} \leq_\Gamma \mathcal{M}^\Gamma$.

(1) Clearly $[\cdot]$ is surjective. Suppose that w sees v in $\mathcal{M}$. Then $[w]$ sees $[v]$ in ${}^{\Gamma}\mathcal{M}$ by the definition of left filtration. So $[w]$ sees $[v]$ in $\mathcal{N}$ since ${}^{\Gamma}\mathcal{M} \leq_{\Gamma} \mathcal{N}$.

(2) Suppose that $w \in V(p)$. Then

$$\begin{aligned} w \in V(p) &\implies [w] \in {}^{\Gamma}V(p) && \text{by the left filtration} \\ &\implies [w] \in V'(p) && \text{by } \leq_{\Gamma} \\ &\implies [w] \in V^{\Gamma}(p) && \text{by } \leq_{\Gamma} \\ &\implies w \in V(p) \end{aligned}$$

So $w \in V(p)$ iff $[w] \in V'(p)$.

(3) Suppose that $[w]$ sees $[v]$ and let $\Box\varphi \in \Gamma$ be such that $w \Vdash \Box\varphi$. Since $[w]$ sees $[v]$ in $\mathcal{N}$, $[w]$ sees $[v]$ in $\mathcal{M}^{\Gamma}$. So by definition of accesibility in $\mathcal{M}^{\Gamma}$, $v \Vdash \varphi$.

□

Definition 5.18. *Let $\mathcal{M} = (W, R, V)$ be a model. We say that $\mathcal{M}$ is* separated *if and only if whenever $w \neq v$, there is some φ such that $w \Vdash \varphi$ and $w' \nVdash \varphi$.*

Example 5.19. *For any normal modal logic S, $(\hat{\mathcal{S}}, V)$ is separated. In addition whenever $f : \mathcal{M} \to \mathcal{N}$ is a normal Γ-filtration, $\mathcal{N}$ is separated.*

6. Finite Model Property

Let $\mathrm{Fin}(S)$ be the class of all finite frames on which S is valid.

Definition 6.1. *Let S be a normal modal logic. We say that S has the* finite model property *if and only if $S = \mathrm{Log}(Fin(S))$.*

Like canonicity, having the finite model property is a sufficient condition on being Kripke complete.

Theorem 6.2. *If S has the finite model property, then S is Kripke complete.*

Proof. It suffices to show that $\mathrm{Log}(\mathrm{Mod}(S)) \subseteq S$. But clearly $\mathrm{Log}(\mathrm{Mod}(S)) \subseteq \mathrm{Log}(\mathrm{Fin}(S))$ and so this follows immediately from the assumption that S has the finite model property. □

Our immediate goal will be to show that all of the standard systems have the finite model property. Note that this shows that to a certain extent the expressive power of the modal language is somewhat limited: we cannot write anything down that, in the standard systems, would be true on all and only infinite models satisfying those systems.

Proposition 6.3. *The trivial system K has the finite model property.*

Proof. It suffices to show that if $\nvdash_K \varphi$ then there is some finite frame $\mathcal{F}$ such that $mathcal F \nVdash \varphi$. So suppose that $\nvdash \varphi$. Since K is Kripke complete, there is some frame $\mathcal{F} = (W, R)$ such that $\mathcal{F} \nVdash \varphi$. So for some model $\mathcal{M} = (W, R, V)$ and world $w \in W$, $\mathcal{M}, w \nVdash \varphi$. Let Γ be the collection of all subformulas of φ and let $f : \mathcal{M} \to \mathcal{N}$ be any normal filtration. By the filtration lemma, $\mathcal{N}, f(w) \nVdash \varphi$. Thus φ is not valid on the underlying frame of $\mathcal{N}$. Moreover, since Γ was finite, $\mathcal{N}$ is also finite and so we are done. □

Proposition 6.4. *Let $f : \mathcal{M} \to \mathcal{N}$ be a surjective morphism.*

(i) If $\mathcal{M}$ is serial, then $\mathcal{N}$ is serial.

(ii) If $\mathcal{M}$ is reflexive, then $\mathcal{N}$ is reflexive.

Proof. Suppose that $\mathcal{M}$ is serial and let $f(w) \in \mathcal{M}$. Then w sees some $v \in \mathcal{M}$. Hence $f(w)$ sees $f(v)$. Since f is surjective it follows that $\mathcal{N}$ is serial.

Suppose that $\mathcal{M}$ is reflexive and let $f(w) \in \mathcal{M}$. Then w sees w and so $f(w)$ sees $f(w)$. Since f is surjective it follows that $\mathcal{N}$ is serial. □

Corollary 6.5. KD *and* KT *have the finite model property.*

Proposition 6.6. *Let $\mathcal{M}$ be a model. Then $\mathcal{M}$ is symmetric only if $^\Gamma\mathcal{M}$ is, for any subformula closed set Γ*

Proof. Suppose that $[w]$ sees $[v]$. Then for some $w' \in [w]$ and $v' \in [v]$, w' sees v'. Since $\mathcal{M}$ is symmetric, v' sees w'. Since $[\cdot]$ is a morphism, $[v']$ sees $[w']$. Since $[w] = [w']$ and $[v] = [v']$, $[v]$ sees $[w]$. □

Corollary 6.7. KB, KBD *and* KTB *have the finite model property.*

Definition 6.8 (Lemmon Filtration). *Let $\mathcal{M} = (W, R, V)$ be a transitive model and Γ a subformula closed set of formulas. We define ${}^{\Gamma}_{L}\mathcal{M} = (W/\sim_\Gamma, R', V')$ to be the structure*

- $[w] \longrightarrow [v]$ *if and only if for all $\Box\varphi \in \Gamma$, and all $w' \in [w]$ and $v' \in [v]$*

$$w' \Vdash \Box\varphi \implies v' \Vdash \varphi \wedge \Box\varphi$$

- $[w] \in V'(p)$ *if and only if there exists $w' \in [w]$ such that $w' \in V(p)$*

Lemma 6.9. *The canonical map*

$$[\cdot] : \mathcal{M} \to {}^{\Gamma}_{L}\mathcal{M}$$

is a filtration. Moreover, if $\mathcal{M}$ is transitive, so is ${}^{\Gamma}_{L}\mathcal{M}$.

Proof. (1) Clearly $[\cdot]$ is surjective. To show that $[\cdot]$ is a morphism, suppose that w see v. Let $\varphi \in \Gamma$, $w' \in [w]$ and $v' \in [v]$ such that $w' \Vdash \Box\varphi$. Then

$$\begin{aligned} w' \Vdash \Box\varphi &\implies w \Vdash \Box\varphi \\ &\implies w \Vdash \Box\Box\varphi \\ &\implies v \Vdash \varphi \wedge \Box\varphi \\ &\implies v' \Vdash \varphi \wedge \Box\varphi \end{aligned}$$

Thus $[w]$ sees $[v]$

Let $p \in \Gamma$. Then

$$\begin{aligned} w \Vdash p &\iff w \in V(p) \\ &\iff [w] \in V'(p) \\ &\iff [w] \Vdash p \end{aligned}$$

Suppose that $\Box\varphi \in \Gamma$ and let w and v be arbitrary such that $[w]$ sees $[v]$. Then

$$\begin{aligned} w \Vdash \Box\varphi &\implies v \Vdash \varphi \wedge \Box\varphi \\ &\implies v \Vdash \varphi \end{aligned}$$

Suppose that $[x] \to [y] \to [z]$. Let $\Box\varphi \in \Gamma$, $x' \in [x], y' \in [y], z' \in [z]$. Suppose that $x' \Vdash \Box\varphi$. Then

$$\begin{aligned} x' \Vdash \Box\varphi &\implies y' \Vdash \varphi \wedge \Box\varphi \\ &\implies y' \Vdash \Box\varphi \\ &\implies z' \Vdash \varphi \wedge \Box\varphi \end{aligned}$$

Thus $[w] \to [z]$ as desired. $\Box$

Corollary 6.10. $\mathsf{K4}, \mathsf{KD4}, \mathsf{KT4}$ *have the finite model property.*

Definition 6.11 (B4 Filtration)**.** *Let $\mathcal{M}$ be a model. We define ${}^{\Gamma}_{B4}\mathcal{M}$ so that the worlds are equivalence classes under agreement on Γ, the valuation is defined as in the left filtration, and $[w]$ sees $[v]$ if and only if for all $\Box\varphi \in \Gamma$, $w' \in [w]$ and $v' \in [v]$, $w' \Vdash \Box\varphi$ only if $v' \Vdash \varphi \wedge \Box\varphi$, and $v' \Vdash \Box\varphi$ only if $w' \Vdash \varphi \wedge \Box\varphi$.*

Lemma 6.12. *The canonical map*

$$[\cdot] : \mathcal{M} \to {}^{\Gamma}_{B4}\mathcal{M}$$

is a filtration. Moreover, if $\mathcal{M}$ is symmetric and transitive, so is ${}^{\Gamma}_{B4}\mathcal{M}$.

Proof. Exercise $\square$

Corollary 6.13. *Let S be axiomatized by some combination of* $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{B4}$*. Then S has the finite model property.*

Corollary 6.14. $\mathsf{S5}$ *has the finite model property since* $\mathsf{S5} = \mathsf{KTB4}$

Definition 6.15 (45 filtration)**.** *Let $\mathcal{M}$ be a model and Γ a subformula closed set of formulas. We define a quotient model ${}^{\Gamma}_{45}\mathcal{M}$ by defining accessibility so that $[w]$ sees $[v]$ if and only if*

$$w' \Vdash \Box\varphi \implies v' \Vdash \varphi \wedge \Box\varphi \text{ and } w' \Vdash \Box\varphi \iff v' \Vdash \Box\varphi$$

for all $w' \in [w]$, $v' \in [v]$ and $\Box\varphi \in \Gamma$.

Lemma 6.16. *The canonical map*

$$[\cdot] : \mathcal{M} \to {}^{\Gamma}_{45}\mathcal{M}$$

is a filtration. Moreover, if $\mathcal{M}$ is transitive and euclidean, so is ${}^{\Gamma}_{B4}\mathcal{M}$.

Proof. Exercise $\square$

Corollary 6.17. $\mathsf{K}, \mathsf{KD45}$ *have the finite model property.*

Definition 6.18 (5 filtration)**.**

Corollary 6.19. $\mathsf{K5}$ *and* $\mathsf{KD5}$ *have the finite model property.*

7. The Finite Model Property Without Canonicity

This section will be concerned with a particular modal logic: Godel-Lob logic. This modal logic is given by

$$\mathsf{GL} = \mathsf{K} + \Box(\Box p \to p) \to \Box p$$

Here the axiom $\Box(\Box p \to p) \to \Box p$ is usually called L. The logic GL will turn out to have the finite model property, and so be Kripke-complete. But it is not canonical, as we will show momentarily. Earlier, we showed that $L(p)$ is valid on a frame (W, R) if and only if R is transitive and well founded. If we already had shown that it was Kripke complete, we could then immediately conclude that $\mathsf{GL} = \mathsf{KL4}$ from correspondence theory. Since we haven't we'll prove this syntactically:

Lemma 7.1. $\vdash_{\mathsf{GL}} \Box p \to \Box\Box p$

Proof.

$$\vdash_{\mathsf{GL}} \Box(p \wedge \Box p) \to \Box p \wedge \Box\Box p$$

$$\vdash_{\mathsf{GL}} \Box(p \wedge \Box p) \to \Box p \text{ and } \vdash_{\mathsf{GL}} \Box(p \wedge \Box p) \to \Box\Box p \qquad \dagger$$

$$\vdash_{\mathsf{GL}} p \to (\Box(p \wedge \Box p) \to p \wedge \Box p)$$

$$\vdash_{\mathsf{GL}} \Box p \to \Box(\Box(p \wedge \Box p) \to p \wedge \Box p)$$

$$\vdash_{\mathsf{GL}} \Box p \to \Box(p \wedge \Box p) \qquad \text{using } L$$

$$\vdash_{\mathsf{GL}} \Box p \to \Box\Box p \qquad \text{by } \dagger$$

□

Let $\hat{\mathcal{G}}$ be the canonical model for GL. Let v be some element not in $\hat{\mathcal{G}}$ and define $\hat{\mathcal{G}}[v]$ to be the structure that results from adding v to $\hat{\mathcal{G}}$ and extending the definition of accessibility so that v sees every element of $\hat{\mathcal{G}}$ while leaving everything else in place.

Claim 7.2. *For all $w \in \hat{\mathcal{G}}$, $\hat{\mathcal{G}}, w \Vdash \varphi$ iff $\hat{\mathcal{G}}[v], w \Vdash \varphi$*

Proof. Note that this valuation is defined the same in both structures, the variables and truth functional operations are trivial. For the necessity operator, we have that for $w \in \hat{\mathcal{G}}$ w sees the same things in both $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}[v]$ □

Claim 7.3. *$\hat{\mathcal{G}}[v], v \Vdash \Box\varphi$ if and only if $\hat{\mathcal{G}} \Vdash \varphi$.*

Proof. Since v sees every world in $\hat{\mathcal{G}}$ this follows immediately. □

Claim 7.4. $\hat{\mathcal{G}}[v] \Vdash \mathsf{GL}$

Proof. By previous claims, we need only check that it holds at v. So suppose that $v \Vdash \Box(\Box\varphi \to \varphi)$. Then

$$\begin{aligned} v \Vdash \Box(\Box\varphi \to \varphi) &\implies \hat{\mathcal{G}} \Vdash \Box\varphi \to \varphi \\ &\implies \hat{\mathcal{G}} \Vdash \Box(\Box\varphi \to \varphi) \\ &\implies \hat{\mathcal{G}} \Vdash \Box\varphi \\ &\implies \hat{\mathcal{G}} \Vdash \varphi \\ &\implies v \Vdash \Box\varphi \end{aligned}$$

□

Lemma 7.5. *$\vdash_{\mathsf{GL}} \Box\varphi_1 \vee \cdots \vee \Box\varphi_n$ only if $\vdash_{\mathsf{GL}} \varphi_i$ for at least one i.*

Proof. Suppose that $\vdash_{\mathsf{GL}} \Box\varphi_1 \vee \cdots \vee \varphi_n$. Then by the above claim we have

$$\begin{aligned} \hat{\mathcal{G}}[v] \Vdash \Box\varphi_1 \vee \cdots \vee \Box\varphi_n &\implies v \Vdash \Box\varphi_1 \vee \cdots \vee \Box\varphi_n \\ &\implies \exists i v \Vdash \Box\varphi_i \\ &\implies \hat{\mathcal{G}} \Vdash \varphi_i \\ &\implies \vdash_{\mathsf{GL}} \varphi_i \end{aligned}$$

□

Lemma 7.6. *There is a world $X \in \hat{\mathcal{G}}$ that sees every other world in $\hat{\mathcal{G}}$.*

Proof. Let $\Gamma = \{\neg\Box\varphi \mid \nvdash_{\mathsf{GL}} \varphi\}$. We want to show that Γ is consistent. For the sake of contradiction, suppose not. Then there exists $\neg\Box\varphi_1, \dots, \neg\Box\varphi_n \in \Gamma$ such that

$$\vdash_{\mathsf{GL}} \neg\Box\varphi_1 \wedge \cdots \wedge \neg\Box\varphi_n \to \bot$$

$$\implies \vdash_{\mathsf{GL}} \Box\varphi_1 \vee \cdots \vee \Box\varphi_n$$

$$\implies \exists i \vdash_{\mathsf{GL}} \varphi_i$$

That's a contradiction. So Γ is consistent. Thus there is some maximally consistent $X \in \hat{\mathcal{G}}$ such that $\Gamma \subseteq X$. So

$$\nvdash_{\mathsf{GL}} \varphi \implies \neg\Box\varphi \in \Gamma$$

$$\implies \neg\Box\varphi \in X\$ \qquad \implies \Box\varphi \notin X$$

Contraposing:

$$\Box\varphi \in X \implies \vdash_{\mathsf{GL}} \varphi \implies \varphi \in Y \ \forall Y \in \hat{\mathcal{G}}$$

By the definition of accessibility in $\hat{\mathcal{G}}$, X sees every Y. □

Theorem 7.7. GL *is not canonical.*

Proof. Let X sees every $Y \in \hat{\mathcal{G}}$. Then X sees X and so $\hat{\mathcal{G}}$ is not well-founded. By correspondence theory, $\hat{\mathcal{G}} \nVdash \mathsf{GL}$. □

Sp GL is not well founded. Is it Kripke complete? We can show the affirmative by checking our other sufficient condition on Kripke completeness: the finite model property.

Definition 7.8. *Let $\Pi = (W, R)$ be a frame. We say that Π is a* finite rooted tree *if*

(i) Π is finite;

(ii) R is transitive and irrreflexive;

(iii) If $R(x, z)$ and $R(y, z)$ then either $R(x, y)$, $x = y$ or $R(y, x)$

(iv) There is a root, i.e. an element x such that if $y \neq x$, then $R(x, y)$.

Remark 7.9. *let $\mathcal{F}$ be a finite rooted tree. Then $\mathcal{F}$ is well-founded. So by correspondence theory $\mathcal{F} \Vdash \mathsf{GL}$.*

Let $\mathbb{F}$ be the class of all finite rooted trees. Since $\mathbb{F} \Vdash \mathsf{GL}$ we know that $\mathsf{GL} \subseteq \mathrm{Log}(\mathbb{F})$. To show the finite model property, we must go in the other direction and show $\mathrm{Log}(\mathbb{F}) \subseteq \mathsf{GL}$. We will prove this contrapositively: we will show that if $\nvdash_{\mathsf{GL}} \varphi$, then there exists $\Pi \in \mathbb{F}$ such that $\Pi \nVdash \varphi$.

So let φ be a formula such that $\nvdash_{\mathsf{GL}} \varphi$.

Lemma 7.10. *There is an element $x \in \hat{\mathcal{G}}$ such that $x \Vdash \neg\varphi \wedge \Box\varphi$.*

Proof. For the sake of contradiction, suppose not. Then

$$
\begin{aligned}
&\hat{\mathcal{G}} \Vdash \Box\varphi \to \varphi \\
&\qquad \Longrightarrow \hat{\mathcal{G}} \Vdash \Box(\Box\varphi \to \varphi) \\
&\qquad \Longrightarrow \hat{\mathcal{G}} \Vdash \Box\varphi \\
&\qquad \Longrightarrow \hat{\mathcal{G}} \Vdash \varphi \qquad\qquad \Longrightarrow \vdash_{\mathsf{GL}} \varphi
\end{aligned}
$$

This contradicts our assumption that $\nvdash_{\mathsf{GL}} \varphi$. ☐

Let Γ be the set of subformulas of φ. Say that a world $y \in \hat{\mathcal{G}}$ is *relevant* if $y \Vdash \neg\psi \wedge \Box\psi$ for some $\psi \in \Gamma$. Say that a chain $x_1, \ldots, x_n$ is acceptable if each term x_i is acceptable.

Lemma 7.11. *Let $\bar{x}$ be an acceptable chain. Then the length of $\bar{x}$ is less than or equal to the cardinality of Γ.*

Proof. Let $\psi_1, \ldots, \psi_n$ witness the relevance of $\bar{x}$ (i.e. $x_i \Vdash \neg\psi_i \wedge \Box\psi_i$). Suppose for the sake of contradiction that the length of $\bar{x}$ is greater than that of Γ. Then for some i and j, $\psi_i = \psi_j$. Since $\hat{\mathcal{G}}$ is transitive, either x_i sees x_j or else x_j sees x_i. Either way we get a contradiction. ☐

Let $\mathbf{x} \in \hat{\mathcal{G}}$ be a selected world with the property that $\hat{\mathcal{G}}, \mathbf{x} \Vdash \neg\varphi \wedge \Box\varphi$. We are going to construct a frame by taking certain extensions of $\mathbf{x}$, and extensions of those extensions, as our worlds. At the first stage, we search through the subformulas of φ for a formula ψ with $\mathbf{x} \Vdash \neg\Diamond\psi$. Let X be the set of all such pairs. For any pair $(\bar{x}, \psi)$, we then search through the set of all $y \in R(\bar{x})$ with $y \Vdash \neg\psi \wedge \Box\psi$. We select on such y and add the sequence $\bar{x}y$ to our frame. At the next stage, we repeat this process but now with y in place of $\bar{x}$. Thus we search through subformulas ψ' such that $y \Vdash \neg\Diamond\psi'$. We then add $\bar{x}yy'$ for $y' \in R(y)$ such that $y' \Vdash \neg\psi' \wedge \Box\psi'$. If no such y' can be found, we stop.

Here is a more formal description of the procedure.

(1) Let $B = \{\mathbf{x}\}$, the singleton of our selected element.

(2) Let $X_n = \{(\bar{x}, \psi) \mid \bar{x} \Vdash \Diamond\neg\psi, f(\bar{x}) \in B_n, \psi \in \Gamma\}$ and for each pair $(\bar{x}, \psi)$ let $h(\bar{x}, \psi)$ be a $y \in R(x_n)$ with $y \Vdash \neg\psi \wedge \Box\psi$ if there is such a y; otherwise $h(\bar{x}, \psi)$ is the empty sequence (). Then $B_{n+1} = \{\bar{x}h(\bar{x}, \psi) \mid (\bar{x}, \psi) \in X_n\}$.

(3) Finally let

$$B = \bigcup_{i=1}^{|\Gamma|} B_i$$

Let $\mathcal{B} = (B, <)$ be a frame such that $\bar{x} < \bar{y}$ if $\bar{x}$ is a property initial segment of $\bar{y}$.

Lemma 7.12. *$\mathcal{B}$ is a finite rooted tree such that every world is an acceptable chain.*

Proof. $\mathcal{B}$ is clearly finite and the relation of being a proper initial segment of is clearly transitive and irreflexive. Moreover if s and r are proper initial segments of t, then either s is r, or one is a proper initial segment of the other. Finally $\mathbf{x}$ is the unique root of the tree: if the construction terminates at B_1 is satisfies this property vacuously. If it continues, clearly by construction $\mathbf{x}$ is an initialy segment of everything that follows and nothign that follows is an initial segment of $\mathbf{x}$. By induction, each $s \in B$ is is acceptable. For if s is acceptable and $y \Vdash \neg\psi \wedge \Box\psi$ for some $\psi \in \Gamma$, then sy is acceptable. $\Box$

Lemma 7.13. *Let $(\hat{\mathcal{G}}, V)$ be the canonical model and define a model $(\mathcal{B}, V')$ such that $\bar{x} \in V'(p)$ if and only if $x_n \in V(p)$ for $p \in \Gamma$. Then*

$$(\mathcal{B}, V')\bar{x} \Vdash \varphi \iff (\hat{\mathcal{G}}, V), x_n \Vdash \varphi$$

for $\varphi \in \Gamma$ *and* $\bar{x} \in B$.

Proof. By induction. Mostly trivial until $\Box$. Suppose that $\bar{x} \Vdash \Box\varphi$ and $x_n \Vdash \neg\Box\varphi$. Thus $x_n \Vdash \Diamond\neg\psi$. So there is a $y \in R(x_n)$ with $y \Vdash \neg\varphi \wedge \Box\varphi$. Note that then we have $x_n \Vdash \Diamond\neg\varphi$ and $y \Vdash \neg\varphi \wedge \Box\varphi$. Hence by construction $\bar{x}h(\bar{x},\varphi) \in B$ with $h(\bar{x},\varphi) \Vdash\!\Vdash \neg\varphi \wedge \Box\varphi$. Thus $h(\bar{x},\varphi) \Vdash \neg\varphi$. Thus by induction $\bar{x}h(\bar{x},\psi) \Vdash \neg\varphi$. But since $\bar{x} < \bar{x}h(\bar{x},\psi)$ and $\bar{x} \Vdash \Box\varphi$, this is a contradiction.

Going in the other direction, suppose that $x_n \Vdash \Box\varphi$. Let $\bar{x} < \bar{y}$ be arbitrary. Then $y_m \in R(x_n)$, where y_m is the last term of $\bar{y}$. So $y_m \Vdash \varphi$. By induction $\bar{y} \Vdash \varphi$. Since $\bar{y}$ was an arbitrary extension of $\bar{x}$, $\bar{x} \Vdash \Box\varphi$. $\Box$

Corollary 7.14. *Suppose that* $\nvdash_{\mathsf{GL}} \varphi$. *Then* $(\mathcal{B}, V', \{\mathbf{x}\}) \Vdash \neg\varphi$, *where* $\mathbf{x}$ *is a point in the canonical model witnessing* $\neg\varphi$, *and the frame* $\mathcal{B}, V'$ *is constructed as above.*

Corollary 7.15. GL *has the finite model property.*